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ALGEBRAIC THEORIES

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CONTENTS

Notation.

$\underline{S}$ denotes the category of sets and functions.

C^o denotes the opposite or dual of a category C.

Nat(F, G) denotes the class of natural maps from F to G.

$\mathbb{R}$ denotes the ring of real numbers.

Q denotes the ring of rational numbers.

Algebraic theories will be denoted by capital italics A, B, ...

For an algebraic theory A : -

A_1	the fundamental object.
A_S	the coproduct $\coprod_S A_1$
$\Omega_S(A)$	the set of S-ary operations, i.e. $\mathrm{Hom}_A(A_1, A_S)$.
$Z(A)$	the centre of A.
A^b	the category of A-models and homomorphisms.
$[A, A]$	the category of (A, A)-bimodels and homomorphisms.
$\{A\}$	the category of primitively generated (A, A)-bimodels.
U_A	the forgetful functor $A^b \longrightarrow \underline{S}$.
F_A	the free A-model functor $\underline{S} \longrightarrow A^b$.
I_A	the full embedding $A \longrightarrow A^b$.

Introduction.

These notes have their origins in a lecture course given at the Matematisk Institut at Aarhus in the Autumn of 1969. The lecture notes for this course were published as No. 22 in the Aarhus Lecture Notes series, and they contained a large number of mistakes, mostly due to the emphasis I had placed on formalism at the expense of foundational rigour. I am particularly grateful to Professor J. Isbell for pointing out some of these mistakes, and for his helpful letters.

I felt there was a need for notes which treated the topic of universal algebra in a categorical way, aimed at the postgraduate student who is beginning research. There are books on category theory which mention this topic in passing [Maclane 3, Pareigis] and there are books on universal algebra which do not use category theory very much [Cohn, Grätzer], but there are no books (yet) which exploit the simplicity of the categorical approach to universal algebra, pioneered by Lawvere, Linton, Freyd and others.

I could have gone the whole way by developing the topic of universal algebra in terms of monads. For compactness of formalism this is undoubtedly the way to go. However, I believe that the approach given here, based on algebraic theories, is more readily comprehensible to the reader who is accustomed to classical algebra. No treatment of algebraic theories can be complete without mention of monads. Since the theory of monads has been expounded at length in many texts [Eilenberg and Moore, Kleisli, Linton 2, 3, Maclane 3] I have merely given an outline in § 6.

My intention in writing these notes is twofold; first, to provide an introduction to algebraic theories, and second, to bring the reader to a point where he can ask himself hitherto unposed questions which arise rather naturally. It is for this reason that the last sections are left hanging in the air; it is for the reader to carry them further. I have included exercises which I hope will suggest tangential developments for which the text has no room.

In the last few decades categorical methods have been invading more and more branches of mathematics. From the beginning, it was clear that universal algebra was ripe prey. To paraphrase, one could say that universal algebra is the study of algebraic systems defined by operations and by universal sentences which relate these operations. For example, rings are defined by operations called "multiplication', 'addition' etc., and laws relating them e.g., associativity, distributivity. This was the classical approach. The laws were expressed by using variables. In the categorical approach, we would say that (many-sorted) universal algebra was the study of limit preserving functors - we shall not consider systems of quite such generality. Instead of variables we have the categorical notion of product. Instead of laws we have commutative diagrams.

The classical approach suffered from one or two disadvantages. One of these was the distinction between primitive and derived operations. One is often introduced to, say, groups as sets with a binary associative operation, a nullary operation for two-sided unit, and a unary inverse operation. These would be described as primitive operations, and the 3-ary operation

$$(x, y, z) \longrightarrow x^2 yz^{-1} x$$

would be described as a derived operation, because it can be built up in terms of the primitive ones. In an algebraic theory there is not such distinction. All the operations are treated on the same footing, and while it is true that certain collections of operations may generate all the rest, no particular generating collection is singled out. In this sense, algebraic theories are defined independently of any particular presentation.

Classical algebra relies heavily on the use of variables to express complicated expressions. In category theory, the role of a variable is played by an identity map. In an arbitrary category, objects do not have elements, so, if we use variables, we must not imagine that they stand for elements.

There is a subtle point of notation which the reader should be aware of: we shall think of an S-ary operation not as a function of S-variables but as a function of one variable, whose values are S-indexed families. The order of the variables is entirely spurious, and arises from the fact that the sets $\{1, 2, \ldots, n\}$ have a natural order relation on them - this is a matter of psychology and the linearity of our writing system rather than of mathematics.

We shall use two notations. If X and S are sets, and X^S denotes the set of functions from S to X, we shall say that a function $\omega : X^S \longrightarrow X$ is an <u>S-ary operation</u> on X. If ξ is a function from S to X we obtain an element $\omega(\xi)$ in X.

Alternatively, we may wish to emphasize that the function $\xi : S \longrightarrow X$ determines an S-indexed family of elements of X, $x_\sigma = \xi(\sigma)$ for $\sigma \in S$. In that case we denote $\omega(\xi)$ by $\omega^\sigma x_\sigma$. The upper case suffix σ in ω^σ is purely symbolic; it is a dummy suffix telling us what ω is to operate upon. This symbolism has various advantages; we might, for example, have a double suffix notation $x_{\sigma\tau}$, $\sigma \in S$ and $\tau \in T$, so that $\omega^\sigma x_{\sigma\tau}$ would be a T-indexed family of elements of X. If ω is an S-ary operation and θ is a T-ary operation, the statement "ω and θ commute" may be expressed by

$$\omega^{\sigma}\theta^{\tau}x_{\sigma\tau} = \theta^{\tau}\omega^{\sigma}x_{\sigma\tau}$$

for any $S \times T$-indexed family $\{x_{\sigma\tau}\}$.

This notation provides a convenient half way house between the 'functions of many variables' approach and the 'functions of functions' approach.

One of the central themes of these notes is the rings-theories analogy, first suggested by F. W. Lawvere [Lawvere 2]. Suppose that R is a ring (with unit, as all rings will be here, but not necessarily commutative). A left R-module is simply an abelian group with the elements of R acting as unary operations on it, subject to certain rules. It is clear that R-modules are algebraic systems of the type considered in universal algebra. The algebraic theory of left R-modules is uniquely determined by R, and conversely. This suggests that we should think of an algebraic theory as a kind of generalised ring, and the models of the algebraic theory as generalised modules (I am grateful to Jon Beck for this pun). This analogy is very fruitful. Many definitions and theorems generalise from rings/modules to theories/models.

It is worth underlining the advantages of a fundamental and long-established mathematical practice - abuse of language. Without it we would be stuck in a mass of unnecessary precision and a superfluity of significance - concealing symbols and suffixes. For example, in ring theory we use the same symbol for a ring R, for its underlying set, for the free left or right R-module on one generator, or for the free R-bimodule on one generator. It is the latter usage in particular that I shall adopt and generalise. I shall not carry this principle to extremes. Indeed, I may appear unnecessarily fastidious by not omitting at times the symbol for a forgetful functor.

The point I am trying to stress is that precision for its own sake can conflict with the requirements of easy reading. On the other hand, there are some confusing points where precision is vital. For example, a 2-ary operation

$$(x_1, x_2) \longrightarrow f(x_1, x_2)$$

determines a 3-ary operation

$$(x_1, x_2, x_3) \longrightarrow f(x_1, x_2)$$

and it is most important to distinguish between them.

In § 1, *algebraic theory* and *model* of an algebraic theory are defined. We will use coproducts rather than products for notational convenience. Our theories are not small categories - they contain operations of arbitrary arity. Instead of truncating our theories, we consider *bounded theories.* This makes the construction of free models in § 2 a little simpler. All the nasty things that can happen to algebraic theories arise, generally speaking, from unboundedness. A theory is bounded if it is generated by a set of operations (whose arities are therefore bounded by some cardinal). A small point is worth mentioning here; we shall invoke the principle of abuse of language quite often by not distinguishing between a set and its cardinal when it comes to the notion of arity.

In § 3 we pause in the development of the subject to consider some special theories, $\mathbb{1}$, $\mathbb{2}$ and $\underline{S}$ and some special kinds of theories, nullary, affine, unary and annular. In § 4 we study the existence of limits and colimits in algebraic categories. First we construct limits, and, in particular, congruences. Using these we can construct coequalizers. Using these, and the fact that theories are categories with coproducts, we can construct coproducts. The contents of this section must have occurred in many texts; we have adopted a complicated but elementary programme at this stage, rather than use the simpler but more abstract methods of monads.

In § 5 we meet maps of theories, and the adjoint pair of functors they induce between the categories of models. The notion of map of theories is clearly important, yet is relatively unexplored.

§ 6 is a brief outline of the relevance of monads and tripleability. In § 7 we study structure-semantics adjointness very superficially - deeper studies may be found elsewhere.

In § 8 we meet bimodels and tensor products of bimodels, concepts of fundamental importance. In § 9 we study algebras over theories. If A is a theory, an A-algebra is defined to be a monoid in the monoidal category of (A, A)-bimodels. An A-algebra X gives rise to a theory which we denote by the same letter, and a map of theories A $\longrightarrow$ X, which we call an <u>essential</u> map. We show that a map of theories is essential if and only if the associated forgetful functor has a right adjoint.

In § 10 we look at commutative theories. The rings-theories analogy works particularly smoothly here. In § 11 we construct free theories using trees. The construction is rather redious, but necessary if we are to describe theories by means of presentations in terms of generators and relations (i. e. operations and laws), which they generally are in practice. § 12 follows § 4 very closely. We consider congruences on theories, and we note that the construction for coproducts of theories breaks down in the unbounded case. We discuss the semantic interpretation of the product of a family of theories, and the coproduct of a family of bounded theories. § 13 introduces the Kronecker product of bounded theories, and § 14 and § 15 develop ideas suggested by the rings-theories analogy, such as Morita equivalence, and the generalization of the notion of matrix ring.

It has been pointed out by Professor Linton that with our definition of model, an isomorphism of models can induce the identity map on the underlying sets, and yet not be an identity map. This arises from the fact that we have not chosen canonical products, and seems a small price to pay for not doing so. It is as if we distinguished between two identical models when we use different coloured ink for the brackets round n-ples of elements!

The tensor product symbol $\otimes$ has been somewhat overworked in these pages. We use it in the following contexts, each in some sense a specialization or generalization of the others: -

(i) tensor product of bimodels.

(ii) Kronecker product of theories.

(iii) tensor product of algebras over a theory.

(iv) tensor product of models of a commutative theory.

The diagram

$$\begin{tikzcd} \text{i)} \arrow[r, Rightarrow] & \text{iii)} \arrow[r, Rightarrow, shift right] \arrow[d, Rightarrow] & \text{ii)} \arrow[l, Rightarrow, shift right] \\ & \text{iv)} & \end{tikzcd}$$

may be interpreted by translating $\Longrightarrow$ as "is a general case of —".

However, the uses of the symbol $\otimes$ are consistent with each other, in accordance with the principle of abuse of language. In particular, see [Freyd].

§1. Algebraic Theories

Definition 1.1 An algebraic theory (or simply, theory) is a category A, with all coproducts, such that every object is a coproduct of copies of a fundamental object A_1.

This means that every object of A has the form A_S for some set S, where A_S denotes the coproduct of a family of A_1's indexed by S. The object $A_\emptyset$ will be an initial object. If the set S is nonempty, and $\sigma \in S$, we write

$$\delta_\sigma^S : A_1 \longrightarrow A_S$$

for the canonical map into the coproduct of the σ-th factor. To recapitulate the definition of coproduct, if

$$\left\{ A_1 \xrightarrow{f_\sigma} A_T \right\}_{\sigma \in S}$$

is an S-indexed family of maps in A, then there is a unique map

$$A_S \xrightarrow{<f_*>} A_T$$

such that

$$A_1 \xrightarrow{\delta_\sigma^S} A_S \xrightarrow{<f_*>} A_T \;=\; A_1 \xrightarrow{f_\sigma} A_T$$

for each σ in S.

We shall write composition of maps in A in the same order as they appear in the diagrams, so we write the equation above as

$$\delta_\sigma^S < f_* > \;=\; f_\sigma \, .$$

For any function $S \xrightarrow{g} T$ between two sets S and T we have a map

$$A_S \xrightarrow{A_g} A_T$$

in A, defined uniquely by the condition that for all σ in S

$$\delta^S_\sigma \cdot A_g = \delta^T_{g(\sigma)} \, .$$

In this way we get a functor

$$j_A : \underline{S} \longrightarrow A \quad : \quad T \longmapsto A_T \, , \; g \longmapsto A_g,$$

which preserves coproducts. We shall see later that this functor is faithful in all but two cases. We call a map in A of the form A_g <u>function-like</u>.

<u>Definition 1.2</u> If A is a theory and S is a set we call a map

$$A_1 \longrightarrow A_S$$

in A an <u>S-ary operation</u> of A. We will also use the notation

$$\Omega_S(A)$$

for the set $\mathrm{Hom}_A(A_1, A_S)$ of S-ary operations of A. The basic properties of coproducts imply that the set

$$\mathrm{Hom}_A(A_T, A_S)$$

is naturally isomorphic to the set of functions

$$T \longrightarrow \Omega_S(A) \, .$$

A bijection $S \longrightarrow S'$ gives an isomorphism $A_S \longrightarrow A_{S'}$, and hence a bijection $\Omega_S(A) \longrightarrow \Omega_{S'}(A)$. It follows that A is completely determined by the sets $\Omega_S(A)$, choosing one S for each cardinal, and by the rules for composing maps in A.

Notice that if the set S is nonempty, so is the set $\Omega_S(A)$, because it contains δ^S_σ for some element σ of S. However, $\Omega_\emptyset(A)$ may or may not be empty.

<u>Definition 1.3</u> Let A be a theory, and let C be a category with products. An <u>A-model</u> in C is a product preserving functor

$$A^o \longrightarrow C .$$

That is to say, it is a contravariant functor from A to C which takes coproducts in A to products in C. A <u>homomorphism</u> between two A-models ia a natural map.

We shall chiefly be concerned with the case $C = \underline{S}$, in which case we shall talk simply of A-models, rather than of A-models in $\mathcal{S}$.

Suppose that

$$A^o \xrightarrow{X} C$$

is an A-model in C. Then we may identify $X(A_S)$ with $\prod_S X(A_1)$, and we shall do this from hereon without comment. In particular, $X(\delta^S_\sigma)$ may be identified with the projection to the σ-th factor.

$$\prod_S X(A_1) \simeq X(A_S) \longrightarrow X(A_1) .$$

The object $X(A_\emptyset)$ must be terminal in C.

For any S-ary operation of A, and A-model X in C we have a map

$$X(\omega) : \prod_S X(A_1) \longrightarrow X(A_1)$$

which we call the <u>action</u> of ω on X. It should be clear that X is uniquely determined by $X(A_1)$ and by the actions $X(\omega)$. We call $X(A_1)$ the underlying object (or set in the case $C = \underline{S}$) of X. The condition that X be a functor ensures that the actions $X(\omega)$ satisfy certain conditions, which may be described by the collection of commutative diagrams in A.

When $C = \underline{S}$, we will denote the action $X(\alpha)$ of a map

$$A_S \xrightarrow{\alpha} A_T$$

in A, for an A-model X, by

$$\xi \longmapsto \alpha.\xi \qquad\qquad \xi \in \prod_T X(A_1) .$$

This notation is consistent in the sense that of the composite $\alpha'.\alpha$ is defined in A, then

$$(\alpha'.\alpha)\xi = \alpha'.(\alpha.\xi) .$$

In this way we get a notation reminiscent of that of a monoid acting on a set. Since we have categories rather than monoids, multiplication is not always defined; further, the elements ξ do not simply come from the underlying set of the A-model X but from Cartesian powers of it. However, this is a small price to pay for the enormous advantages of this notation over the classical functional notation, which is hard put to it describing anything more complicated than a binary operation. What we have done is adopt a notation whereby operations all stand to the left of their arguments. For example, in the theory of rings, instead of writing

$$a + b \quad \text{or} \quad a \times b$$

we write $+(a,b)$ or $\times(a,b)$. An expression like

$$(a \times b) + (c \times d)$$

would come out $+ . \langle x, x\rangle . (a,b,c,d)$. In this way we can employ the $\langle \ldots \rangle$ notation for coproducts of maps in a theory to denote composite operations.

Let us consider the action of function-like maps in a theory. Let $S \xrightarrow{g} T$ be a function, A a theory, and X an A-model. We obtain a function

$$X(A_g) : \prod_T X(A_1) \longrightarrow \prod_S X(A_1) .$$

The effect of this function is as follows: we may identify an element of $\prod_T X(A_1)$ with a function

$$T \longrightarrow X(A_1) \ ,$$

and this gets taken by $X(A_g)$ to the composite

$$S \xrightarrow{g} T \longrightarrow X(A_1) .$$

For example, if g is injective then $X(A_g)$ has the effect of "forgetting" some of the variables, i.e. those indexed by elements of S not in the image of g. If g is surjective, no variables are forgotten but some may be repeated. If g is a bijection, the variables are simply permuted. In this way, the function-like maps of A perform a useful book-keeping service.

For example, when A is the theory of rings, the 4-ary operation

$$(a,b,c,d) \longmapsto b^2 + ac$$

could be described as

$$+ . \langle x, x \rangle . A_g$$

where $g : \{1,2,3,4\} \longrightarrow \{1,2,3,4\}$ is given by $g(1) = g(2) = 2$, $g(3) = 1$, $g(4) = 3$.

Suppose now that $X \xrightarrow{\theta} Y$ is a homomorphism of A-models. Since θ is a natural map, for each set S and S-ary operation $\omega \in \Omega_S(A)$ we have a commutative diagram

$$\begin{array}{ccc} \prod_S X(A_1) & \xrightarrow{\theta_{A_S}} & \prod_S Y(A_1) \\ {\scriptstyle X(\omega)}\downarrow & & \downarrow{\scriptstyle Y(\omega)} \\ X(A_1) & \xrightarrow{\theta_{A_1}} & Y(A_1) \end{array} .$$

By taking $\omega = \delta^S_\sigma$ for each $\sigma \in S$, we see that θ_{A_S} is just $\prod_S \theta_{A_1}$. It follows that θ is uniquely determined by the map $h = \theta_{A_1}$ and that any map

$$h : X(A_1) \longrightarrow Y(A_1)$$

for which the diagram

$$\begin{array}{ccc} \prod_S X(A_1) & \xrightarrow{\prod_S h} & \prod_S Y(A_1) \\ {\scriptstyle X(\omega)}\downarrow & & \downarrow{\scriptstyle Y(\omega)} \\ X(A_1) & \xrightarrow{h} & Y(A_1) \end{array}$$

commutes for all sets S and $\omega \in \Omega_S(A)$, determines a homomorphism $X \longrightarrow Y$. If $\xi \in \prod_S X(A_1)$ and $X \xrightarrow{\theta} Y$ is a homomorphism we denote the image of ξ under $\prod_S \theta_{A_1}$ by $\xi.\theta$. The commutativity of the diagram above can now be expressed:

$$\omega.(\xi.\theta) = (\omega.\xi).\theta$$

or

"homomorphisms commute with operations" .

A particular consequence of the remarks above is that for any pair of A-models, X, Y the homomorphisms from X to Y are in bijective correspondence with a subset of the set of functions from $X(A_1)$ to $Y(A_1)$. Hence, A-models and their homomorphisms form a category, which we shall denote by A^b.

We will write composition of maps in A^b in the same order as the arrows in diagrams, so that if θ, θ' are composable maps in A^b, then, with the notation above,

$$(\xi.\theta)\theta' = \xi.(\theta.\theta') .$$

The assignments $X \longmapsto X(A_1)$, $\theta \longmapsto \theta_{A_1}$ define a functor

$$U_A : A^b \longrightarrow \underline{S}$$

known as the "forgetful" functor, because it forgets A-model structure. The remarks above about a homomorphism being determined by its underlying function give us the following:-

<u>Proposition 1.4</u> The functor

$$U_A : A^b \longrightarrow \underline{S}$$

is faithful. If θ is a map in A^b such that $U_A(\theta)$ is bijective, then θ is an isomorphism. If T is a set and X is an A-model, then for every bijection $T \xrightarrow{\gamma} U_A(X)$ there is an A-model Z and an isomorphism $Z \xrightarrow{\theta} X$ in A^b such that $U_A(Z) = T$ and $U_A(\theta) = \gamma$.

We leave the proof to the reader. This proposition is usually expressed by saying that U_A reflects and creates isomorphisms.

We shall refer to categories of the form A^b for some algebraic theory A as <u>algebraic</u>.

Exercises 1

1. Express the axiom of distributivity of multiplication over addition as a commutative diagram in the theory of rings.

2. If a, b, c are elements of a group, let

$$\langle a, b, c\rangle = ab^{-1}c.$$

Show that for all a, b, c, d, e the relations

$$\langle a, a, b\rangle = \langle b, a, a\rangle = b$$

$$\langle\langle a, d, c\rangle, b, e\rangle = \langle a, \langle b, c, d\rangle, e\rangle = \langle a, d, \langle c, b, e\rangle\rangle$$

hold.

Is the algebraic theory defined by a 3-ary operation satisfying these laws the theory of groups?

3. Give an example of an operation in the theory of groups of infinite arity.

§2. Free models

Let A be an algebraic theory. For any object A_S in A the functor

$$\mathrm{Hom}_A(-,A_S) : A^0 \longrightarrow \underline{S}$$

is product preserving, and so is an A-model. A map

$$A_S \xrightarrow{\alpha} A_T$$

gives rise to a natural map

$$\mathrm{Hom}_A(-,\alpha) : \mathrm{Hom}_A(-,A_S) \longrightarrow \mathrm{Hom}_A(-,A_T)$$

and so a homomorphism of A-models. In this way we get a functor

$$I_A : A \longrightarrow A^b \qquad\qquad \alpha \longmapsto \mathrm{Hom}_A(-,\alpha) .$$

In §1 we remarked that we had a functor

$$j_A : \underline{S} \longrightarrow A \qquad\qquad g \longmapsto A_g .$$

We denote by

$$F_A : \underline{S} \longrightarrow A^b$$

the composite $\underline{S} \xrightarrow{j_A} A \xrightarrow{I_A} A^b$.

Theorem 2.1 The functor $F_A : \underline{S} \longrightarrow A^b$ is left adjoint to the forgetful functor $U_A : A^b \longrightarrow \underline{S}$.

Proof: Let S be a set and X an A-model. Using the Yoneda lemma, we have the following sequence of natural bijections:

$$
\begin{aligned}
& \mathrm{Hom}_{A^b}(F_A(S), X) = && \text{by definition of } F_A(S) \text{ and definition of homomorphism;}\\
= \; & \mathrm{Nat}\,(\mathrm{Hom}_A(-, A_S), X) && \\
= \; & X(A_S) && \text{by Yoneda lemma;}\\
\cong \; & \prod_S X(A_1) && \text{by definition of A-model;}\\
\cong \; & \mathrm{Hom}_{\underline{S}}(S, U_A(X)) && \text{by definition of } U_A .
\end{aligned}
$$

Let us analyze in more detail the adjoint pair of functors (F_A, U_A). First, note that

$$U_A\, F_A\, (S) = \mathrm{Hom}_A\, (A_1, A_S) = \Omega_S\, (A) .$$

The front adjunction

$$\eta_S : S \longrightarrow U_A\, F_A\, (S)$$

is given by $\sigma \longmapsto \delta_\sigma^S$, for $\sigma \in S$. The end adjunction

$$\epsilon_X : F_A\, U_A\, (X) \longrightarrow X$$

is the unique homomorphism taking $\delta_x^{U_A(X)}$ to x, for $x \in U_A(X)$. It is clear from this description that $U_A(\epsilon_X)$ is surjective for any X.

Suppose that X is an A-model, and that M is a subset of $U_A(X)$. Adjoint to the inclusion function

$$M \longrightarrow U_A\, (X)$$

there will be a unique homomorphism, by theorem 2.1,

$$F_A(M) \xrightarrow{\mathcal{E}} X .$$

If $U_A(\mathcal{E})$ is surjective, we shall say that the set M <u>generates</u> X. Every A-model X has a generating set ; indeed, the remarks above show that $U_A(X)$ is always a generating set. The point is that there may exist smaller generating sets. If α is a cardinal we shall say that X is <u>α-generated</u> if it has a generating set of cardinality less than α.

If the subset M of $U_A(X)$ generates X, then every element of $U_A(X)$ may be written as $\omega \xi$ where ξ is a family of elements of M. Indeed, the definition shows that we may take ξ to be a certain fixed family, namely the M-indexed set of elements of $U_A(X)$ whose m-th member is m itself. We see that $U_A(\mathcal{E})$ defines a surjection from $\Omega_M(A)$ to $U_A(X)$.

We say that M generates X <u>freely</u> if the expression for a general element of $U_A(X)$ as $\omega \xi$ is unique, i.e. if every element is a unique expression in terms of the generators. This means that $U_A F_A(M) \xrightarrow{U_A(\mathcal{E})} U_A(X)$ must be bijective, and so by lemma 1.4, the map $F_A(M) \xrightarrow{\mathcal{E}} X$ is an isomorphism.

This leads us to make the definition: an A-model is free if it is isomorphic to a representable functor, i.e. one of the form $\mathrm{Hom}_A(-, A_M) = F_A(M)$ for some set M.

We may now restate theorem 1.2 in more familiar terms:-

Let X be an A-model freely generated by a subset M of $U_A(X)$, and let Y be an A-model. Then every function from M to $U_A(Y)$ lifts uniquely to a homomorphism from X to Y.

Recall that we have a functor

$$I_A : A \longrightarrow A^b \qquad \alpha \longmapsto \mathrm{Hom}_A(-, \alpha) .$$

The Yoneda lemma tells us that this functor is full and faithful. This gives us the following:

<u>Theorem 2.2</u> An algebraic theory A is equivalent to the full subcategory of free A-models.

In fact, this provides us with one of the easiest ways of describing an algebraic theory. We have only to know what the free models are to know the theory. Let us look at an example from the theory of groups (let us call it Gp). A homomorphism from a free group on one generator u to a free group on three generators, x, y, z is uniquely determined by the image of u ; suppose it is $x^2 z^{-1} xy$. What we have established so far tells us that this should correspond to a map $(Gp)_1 \longrightarrow (Gp)_3$ in Gp (here 1 and 3 stand for one element and three element sets). But such a map corresponds to a 3-ary operation. Clearly, this is the operation

$$(g_1, g_2, g_3) \longmapsto g_1^2 g_3^{-1} g_1 g_2 \ .$$

It may be convenient to identify the category A with its image in A^b under I_A. In that case we write A_S in place of $F_A(S)$.

Then, if S is a set, an S-indexed family of elements of an A-model X is given by a map

$$A_S \xrightarrow{\xi} X$$

and if $A_T \xrightarrow{\omega} A_S$ is a T-indexed family of S-ary operations, the composite in A^b

$$A_T \xrightarrow{\omega} A_S \xrightarrow{\xi} X$$

clearly gives $\omega\xi$, so our notation is consistent. By this means we can put operations (compose on the left) and homomorphisms (compose on the right) into one category.

In the theory of, say, rings, every element of a ring can be expressed by means of finitary operations on a set of generators of the ring. The analogue of this statement is not true for an arbitrary theory. If α is a cardinal, we shall say that a map $A_T \xrightarrow{\omega} A_S$ in a theory A is <u>α-bounded</u> if it factors through a function-like map $A_M \xrightarrow{A_g} A_S$, where M is a set of cardinality less than α. We shall say that an algebraic theory A is α-bounded if every map in A is α-bounded. Finally, we shall say that an algebraic theory is <u>bounded</u> if it is α-bounded for some cardinal α.

<u>Proposition 2.3</u> If A is an α-bounded theory, and X is an A-model then every element of $U_A(X)$ can be expressed as the result of applying an operation of less than α to a family of generators.

Proof. Let M be a set of generators of X, and let ξ be the M-indexed set of elements of $U_A(X)$ whose m-th member is m. We have seen that every element of $U_A(X)$ is expressible as $\omega\xi$ where $\omega \in \Omega_M(A)$. By assumption, for a given ω there exists a set N of cardinality less than α and a function $h : N \longrightarrow M$ such that $\omega = \omega_1 . A_h$ for some $\omega_1 \in \Omega_N(A)$. Thus $\omega\xi = \omega_1(A_h\xi)$. But $A_h\xi$ is just an N-indexed set of generators.

It is worth remarking that an algebraic theory A is not a small category. For that reason we forbore to speak of the category of set-valued functors on A^o. However, many of the unpleasant consequences of the fact can be avoided by restricting attention to bounded theories. Classical universal algebra generally confined itself to finitary (i.e. $\aleph_0$-bounded) theories.

Note that if the term α-bounded is to make much sense, we must restrict attention to regular cardinals ; that is to say those with the following property: - the cardinal α is <u>regular</u> if given any family T_i $_{i\in I}$ of sets of cardinality less than α indexed by a set I of cardinality less than α then $\bigcup_{i\in I} T_i$ is a set of cardinality less than α.

The cardinals 2 and $\aleph_0$ are regular. The reader is invited to work out for himself what 2-bounded theories must be like.

Exercises 2

1. Let Conv be the category whose objects are finite dimensional closed simplexes, and whose maps are linear maps of one simplex to another. Show that Conv is an algebraic theory, and that any convex subset of a Euclidean space is a Conv-model.

2. Let R^+ denote the extended half real line, i.e. the set of positive real numbers together with a symbol ∞. If $r_1, r_2, \ldots$ is a countable sequence of elements of R^+, define $\sum_1^\infty r_i$ to be the limit of the sequence $r_1, r_1+r_2, r_1+r_2+r_3, \ldots$ if it exists, and ∞ otherwise. Define a theory of "abelian monoids with countable sums" for which R^+ is a model.

3. Let CH denote the category whose objects are Stone-Cech compactifications of discrete spaces, and whose maps are continuous functions. Show that CH is an algebraic theory. Show that CH is not bounded.

4. Formulate the concept of a topological algebraic theory. Show that Conv is a topological theory is a natural way.

§ 3. Some Special Theories.

In any subject there is a struggle between the general and particular. Examples are needed to illustrate and motivate the general theory. The general theory is needed to provide a descriptive language to discuss the examples with. At this point we hold back the tide of abstraction to consider some examples of theories. Generally, speaking, the examples will suggest further definitions and conditions upon theories which generalize the properties holding true for the example in question. These bear with them whole panoplies of mathematical motifs, tedious to write down in detail, but clear enough after one has had a little practice with them.

Example 1. The category $\underline{S}$ of sets and functions is an algebraic theory, since it has coproducts, and every set is a coproduct of 1's, where 1 denotes a fixed singleton set.

Suppose that $F : \underline{S}^0 \longrightarrow \underline{S}$ is an $\underline{S}$-model. Then for any set S,

$$F(S) \simeq F(\coprod_S 1) \simeq \prod_S F(1) \simeq \mathrm{Hom}_{\underline{S}}(S, F(1)),$$

so that F is a free $\underline{S}$-model. Thus

$$I_{\underline{S}} : \underline{S} \longrightarrow \underline{S}^b$$

is an equivalence of categories. Note that

$$j_{\underline{S}} : \underline{S} \longrightarrow \underline{S}$$

is simply the identity functor, so that

$$F_{\underline{S}} : \underline{S} \longrightarrow \underline{S}^b$$

and hence $U_{\underline{S}} : \underline{S}^b \longrightarrow \underline{S}$ is an equivalence of categories. From hereon we shall identify an $\underline{S}$-model with its underlying set. We shall say that $\underline{S}$ is a theory with no nontrivial operations. The only 1-ary operations are the maps δ_t^T for $t \in T$.

Example 2. Let $\mathbb{1}$ denote a category with precisely one map. It clearly has coproducts, and for all sets S we have $\mathbb{1}_S = \mathbb{1}_1$. It follows that $\mathbb{1}$ is a theory. We have seen that for any theory A and A-model X, the set $X(A_\emptyset)$ has to be terminal, i.e. a singleton set. Since $\mathbb{1}_1 = \mathbb{1}_\emptyset$, a $\mathbb{1}$-model is a singleton set.

Example 3. Let $\mathbb{2}$ denote the category with two objects $\mathbb{2}_\emptyset$ and $\mathbb{2}_1$, and precisely one non-identity map, from $\mathbb{2}_\emptyset$ to $\mathbb{2}_1$. Then every object of $\mathbb{2}$ is a coproduct of copies of $\mathbb{2}_1$. To be precise

$$\coprod_S \mathbb{2}_1 = \mathbb{2}_1 \qquad \text{if } S \neq \emptyset$$

$$\coprod_\emptyset \mathbb{2}_1 = \mathbb{2}_\emptyset .$$

If $S \neq \emptyset$, all the maps δ_σ^S agree, so a $\mathbb{2}$-model is either a singleton set or empty.

<u>Theorem 3.1</u> If the theory A is neither $\mathbb{1}$ nor $\mathbb{2}$, then

$$j_A : \underline{S} \longrightarrow A$$

is a faithful functor.

<u>Proof.</u> Suppose j_A is not faithful. Then there exists a set S with at least two elements σ_1, σ_2 such that $\sigma_1 \neq \sigma_2$ but $\delta_{\sigma_1}^S = \delta_{\sigma_2}^S$ in A. Let f and g be T-ary operations of A, and let $\{h_\sigma\}_{\sigma \in S}$ be an S-indexed family of T-ary operations such that $h_{\sigma_1} = f$ and $h_{\sigma_2} = g$. Then:

$$f = \delta_{\sigma_1}^S < h_* > = \delta_{\sigma_2}^S <h_*> = g.$$

Hence, if there is a T-ary operation, it is unique. Thus, $A = \mathbb{2}$ if $\Omega_{\emptyset}(A) = \emptyset$ and $A = \mathbb{1}$ if $\Omega_{\emptyset}(A) \neq \emptyset$.

It follows that, with the two exceptions $\mathbb{1}$ and $\mathbb{2}$, every algebraic theory contains a subcategory equivalent to $\underline{S}$. Just as some ring theorists like to rule out the zero ring, some authors like to rule out $\mathbb{1}$ and $\mathbb{2}$.

Let us look at nullary operations in more detail. A nullary operation of a theory A is an element of $\Omega_{\emptyset}(A)$. The action of a nullary operation $\lambda \in \Omega_{\emptyset}(A)$ on an A-model X is a function

$$X(\lambda) \ : \ X(A_{\emptyset}) \longrightarrow X(A_1).$$

But $X(A_{\emptyset})$ is a singleton set, so $X(\lambda)$ determines an element of $U_A(X)$ (which we usually write simply as λ).

We know from § 2 that $F_A(\emptyset)$, a free A-model on no generators, has the set of nullary operations as its underlying set. Since F_A has a right adjoint it preserves colimits, and so, as $\emptyset$ is an initial object in $\underline{S}$, $F_A(\emptyset)$ is initial in A^b.

<u>Proposition 3.2.</u> The empty set has an A-model structure if and only if A has no nullary operations.

Proof. If A has no nullary operations, then $U_A F_A(\emptyset) = \Omega_{\emptyset}(A) = \emptyset$. Conversely, if X is an A-model such that $U_A(X) = \emptyset$, since there is a (unique) homomorphism $F_A(\emptyset) \longrightarrow X$, it follows that $U_A F_A(\emptyset) = \emptyset$.

Note that for any theory A, the constant functor

$$A^0 \xrightarrow{1} \underline{S}$$

taking the value of a singleton set, is always an A-model, and that this A-model is terminal in A^b.

By a zero object we mean an object that is both initial and terminal.

Proposition 3.3. If A is an algebraic thoery, then A^b has a zero object if and only if A has precisely one nullary operation.

Proof. A^b has a zero object if and only if $F_A(\emptyset) \simeq 1$.

To a nullary operation $A_1 \xrightarrow{\lambda} A_\emptyset$ we may associate, for any set S, an S-ary operation

$$A_1 \xrightarrow{\lambda} A_\emptyset \longrightarrow A_S$$

where $A_\emptyset \longrightarrow A_S$ is the unique map. In classical universal algebra little distinction was drawn between them; here the distinction is vital. Of course, the S-ary operation $A_1 \xrightarrow{\lambda} A_\emptyset \longrightarrow A_S$ has an action given by a constant function taking the value λ . Though the distinction between them may seem pedantic, for the construction below it is obviously important.

For any theory A, let $\bar{A}$ denote the category obtained from A by removing all maps into $A_\emptyset$. To be more precise, $\bar{A}$ has objects $\bar{A}_S$, and maps given by: -

$$\begin{aligned} \mathrm{Hom}_{\bar{A}}(\bar{A}_S, \bar{A}_T) &= \mathrm{Hom}_A(A_S, A_T) \qquad && T \neq \emptyset \neq S \\ &= \emptyset && T = \emptyset . \end{aligned}$$

Compositon of maps in $\bar{A}$ is defined as in A. Because a union of empty sets is empty, it follows that $\bar{A}$ is a theory with no nullary operations. The nullary operations of A still leave their trace. For example, if $S \neq \emptyset$, a composite

$$A_1 \longrightarrow A_\emptyset \longrightarrow A_S$$

gives rise to a map $\bar{A}_1 \longrightarrow \bar{A}_S$, though of course it no longer factorizes through $\bar{A}_\emptyset$. A little thought should convince the reader that $\bar{A}^b$ is obtained from A^b by adjoining a single model, the empty set. For example $\overline{Gp}$ is the theory of groups, with axioms so chosen so as to allow the empty set as a model.

Now we turn our attention to unary operations. For any theory A, the set $\Omega_1(A)$ of unary operations of A, has a natural monoid structure, with composition of operations as multiplication. Conversely, given a monoid G, we may define a 2-bounded theory, which we shall also denote by G, such that $\Omega_\emptyset(G) = \emptyset$ and $\Omega_1(G) = G$. These conditions determine the theory G completely. If $S \neq \emptyset$, an S-ary operation of the theory G is of the form $g.\delta_\sigma^S$ for some $\sigma \in S$ and $g \in \Omega_1(G)$. A G-model is simply a left G-set, and a homomorphism is a G-equivariant function. Theories of this type we call <u>unary</u>. Modulo an abuse of language, unary theories are monoids.

<u>Proposition 3.4</u> If $A \neq \mathbb{1}$ and $\Omega_\emptyset(A) \neq \emptyset$, then $\Omega_1(A)$ is not the trivial monoid.

Proof. Suppose $\lambda \in \Omega_\emptyset(A)$, if $\Omega_1(A)$ is trivial, then

$$A_1 \xrightarrow{\lambda} A_\emptyset \longrightarrow A_1$$

is the identity map of A_1, so $A_\emptyset$ is isomorphic to A_1. Hence $A \simeq \mathbb{1}$, a contradiction.

For any non-empty set S, the unique function $S \longrightarrow 1$ gives a function-like map

$$\Delta : A_S \longrightarrow A_1$$

whose action is to replace a variable x by the constant S-indexed family taking the value x. We call a map $A_S \xrightarrow{\alpha} A_T$ in A affine if

$$A_S \xrightarrow{\alpha} A_T \xrightarrow{\Delta} A_1 = A_S \xrightarrow{\Delta} A_1 .$$

Thus, an S-ary operation is affine if when it acts on a constant S-indexed family of elements equal to x it gives the result x. For example, in Gp, multiplication is not affine, but the 3-ary operation $(g_1, g_2, g_3) \longmapsto g_1 g_2^{-1} g_3$ is.

It follows immediately from the definition that function-like maps are affine, and that a composite of affine maps is affine. In fact the affine maps in a theory A form a subcategory Aff(A), which is easily seen to be a theory in its own right. We call a theory A affine if A = Aff(A).

Proposition 3.5. The following statements imply each other.

(i) $\Omega_1(A)$ is the trivial monoid.

(ii) The free A-model on one generator has only one element.

(iii) A is affine.

We leave the proof as an easy exercise.

Suppose that K is a ring (with unit). Let us denote by Mat(K) the category of all free left K-modules and K-homomorphisms between them. Since every free left K-module is a coproduct of copies of K itself, considered as the free left K-module on one generator, Mat(K) is an algebraic theory. It is clearly a finitary theory, and its models are left K-modules. If S and T are finite sets, $\mathrm{Hom}_{\mathrm{Mat}(K)}(\mathrm{Mat}(K)_S, \mathrm{Mat}(K)_T)$ may be identified with the set of SxT - matrices with coefficients in K. Composition of maps corresponds to matrix multiplication.

An n-ary operation of Mat(K) is given by an n-ple $(k_1, \ldots , k_n)$ of elements of K, and its action upon a left K-module M is given by

$$(m_1, \dots, m_n) \longrightarrow k_1 m_1 + \dots + k_n m_n .$$

We shall call a theory of the form Mat(K) for some ring K annular. The holy principle of abuse of language suggests that we abbreviate Mat(K) to simply K, using the same symbol for both ring and theory. Thus $\mathbb{Z}$ denotes the theory of abelian groups as well as the ring of integers.

The symbolic equation

$$\text{rings/modules} \sim \text{theories/models}$$

will serve as an inspiration both for our terminology and for the questions we ask ourselves about theories.

I am indebted to Jon Beck for the model-module pun. At this point the alert reader could develop most of the rest of the book for himself. The reader is limited to check for himself the consequences of any new definition or theorem for the special class of annular theories, as a source of illumination.

Exercises 3

1. Show that $\mathcal{2} \simeq \overline{\mathbb{1}}$

2. Vrite down operations and laws between them defining $\overline{\mathrm{Gp}}$.

3. Describe $\mathrm{Aff}(\mathbb{Z})$ and its models.

4. Show that an annular theory is a category with finite limits.

5. If G is a unary theory, show that U_G has a right adjoint as well as a left adjoint.

6. If A is a theory for which U_A has a right adjoint, show that A is unary.

§ 4. The completeness of algebraic categories.

In this chapter we will show that for any algebraic theory A, the category A^b of A-models is complete and cocomplete, i.e. that any diagram in A^b has a limit and a colimit. The chapter splits naturally into two parts; in the first we will deal with limits, and then in the second, using material from the first part, explicitly the notion of congruence, we will deal with colimits.

Readers who are more interested in panoramas than close-ups are advised to skip this chapter as it is long and technical. Were it not that some of the techniques will be needed in later chapters, it would have been relegated to an appendix.

(i) Limits

Because the forgetful functor

$$U_A : A^b \longrightarrow \underline{S}$$

has a left adjoint it must preserve any limits which exist in A^b. This tells us that in order to construct the limit of a diagram in A^b we must try to endow the limit of the underlying diagram in $\underline{S}$ with an A-model structure. Suppose I is a small category and that

$$E : I \longrightarrow A^b$$

is a functor. Let L be the limit of the functor

$$I \xrightarrow{E} A^b \xrightarrow{U_A} \underline{S}$$

with projection maps $p_i : L \longrightarrow U_A(E(i))$ for each $i \in I$. We wish to put an A-model structure on the set L so that the functions p_i define homomorphisms.

We define an A-model X as follows:-

$$X(A_S) = \prod_S L \quad .$$

For any map $A_S \xrightarrow{\alpha} A_T$ in A, we define $X(\alpha)$ to be the composite

$$\prod_T L \xrightarrow{\simeq} \varprojlim_i \prod_T U_A E(i) \xrightarrow{\varprojlim E(i)(\alpha)} \varprojlim_i \prod_S U_A E(i) \xrightarrow{\simeq} \prod_S L$$

where the outer maps are the canonical isomorphisms stating that limits commute with products. It is immediate to verify that X is an A-model, that $U_A(X) = L$, that the projections p_i define homomorphisms $X \longrightarrow E(i)$, and that they make X a limit of the functor E.

If $X \xrightarrow{i} Y$ is a homomorphism of A-models such that $U_A(i)$ is the inclusion of $U_A(X)$ as a subset of $U_A(Y)$ we call X a <u>submodel</u> of Y and write

$$X \subseteq Y.$$

If $\{Z_\nu\}$ is a family of submodels of an A-model Y, then the joint pullback of the inclusions $Z_\nu \subseteq Y$ is again a submodel, which we denote by [illegible] and call the <u>intersection</u> of the family $\{Z_\nu\}$. Of course, we have

$$U_A\left(\bigcap_\nu Z_\nu\right) = \bigcap_\nu U_A(Z_\nu)$$

If $S \subseteq U_A(X)$, for X an A-model, we may form the intersection of all the submodels of X whose underlying sets contain S. We call this intersection $\langle S \rangle$, the submodel <u>generated</u> by S. If $\langle S \rangle = X$ we say that S is a set of generators of X. This agrees with the terminology for free models introduced in § 2, as proposition 4.1. below indicates. If $X \xrightarrow{\phi} Y$ is an arbitrary homomorphism of A-models, the subset

$\mathrm{Im}U_A(\phi)$ of $U_A(Y)$ carries a natural A-model structure, making it a submodel of Y which we denote by $\mathrm{Im}\phi$. We obtain a factorization of ϕ

$$X \xrightarrow{p} \mathrm{Im}\,\phi \xrightarrow{q} Y$$

where $U_A(p)$ is surjective and $U_A(p)$ injective.

Proposition 4.1. If $F_A(S) \xrightarrow{\phi} X$ is the homomorphism adjoint to the inclusion $S \subseteq U_A(X)$, then

$$\mathrm{Im}\,\phi = \langle S \rangle .$$

We leave the proof as an easy exercise.

At this point we remind the reader of two simple categorical notions:

Definition 4.2. An equivalence relation in a category C is a jointly monic pair

$$K \underset{k_1}{\overset{k_0}{\rightrightarrows}} L$$

so that for any X in C,

$$\mathrm{Hom}_C(X,K) \xrightarrow{\langle \mathrm{Hom}_C(X,k_0),\ \mathrm{Hom}_C(X,k_1)\rangle} \mathrm{Hom}_C(X,L) \times \mathrm{Hom}_C(X,L)$$

describes an equivalence relation on $\mathrm{Hom}_C(X,L)$. If C has finite limits we shall abuse language and call the single map

$$K \xrightarrow{\langle k_0,\, k_1\rangle} L \times L$$

an equivalence relation. We may, of course, describe an equivalence relation internally in this case, by means of pull-backs and the commutativity of certain diagrams. It follows that a functor which preserves finite left limits also preserves equivalence relations. Hence, in particular, an equivalence relation on an A-model X is given by a submodel $\Gamma \subseteq X \times X$ such that $U_A(\Gamma)$ is an equivalence relation on $U_A(X)$. The usual word for an equivalence relation in an algebraic category is a congruence.

Proposition 4.3. An intersection of congruences is a congruence.

This follows from the fact that an intersection of equivalence relations is an equivalence relation.

Now we come to the second categorical concept: the kernel pair of a map $L \xrightarrow{f} M$ in a category is a pair of maps

$$K \underset{d_1}{\overset{d_0}{\rightrightarrows}} L$$

such that the diagram

$$\begin{tikzcd} K \arrow[r, "d_0"] \arrow[d, "d_1"'] & L \arrow[d, "f"] \\ L \arrow[r, "f"] & M \end{tikzcd}$$

is a pullback. Any category with finite limits has kernel pairs.

Proposition 4.4. Kernel pairs are equivalence relations.

We leave the proof as an exercise for the reader. We wish to show that the converse holds in an algebraic category. Suppose that $\Gamma \subseteq X \times X$ is a congruence on the A-model X. Denote by $U_A(X)/U_A(\Gamma)$ the set of $U_A(\Gamma)$ - equivalence classes, and by $[x]$ the $U_A(\Gamma)$-equivalence class containing x. Thus

$$[x] = [x']$$

if and only if $(x, x') \in U_A(\Gamma)$.

Let $\{(x_\sigma, x'_\sigma)\}_{\sigma \in S}$ be an S-indexed family of elements of $U_A(\Gamma)$, and let $\omega \in \Omega_S(A)$. Since Γ is a submodel of $X \times X$, it follows that

$$\omega^\sigma(x_\sigma, x'_\sigma) = (\omega^\sigma x_\sigma, \omega^\sigma x'_\sigma)$$

In other words, if for all $\sigma \in S$, $[x_\sigma] = [x'_\sigma]$ then $[\omega^\sigma x_\sigma] = [\omega^\sigma x'_\sigma]$.

Hence we may define an A-model X/Γ by $U_A(X/\Gamma) = U_A(X)/U_A(\Gamma)$ and by defining $(X/\Gamma)(\omega)$ by the formula

$$\{[x_\sigma]\}_{\sigma \in S} \longmapsto [\omega^\sigma x_\sigma]$$

It is clear that the projection $x \longrightarrow [x]$ defines a homomorphism

$$X \longrightarrow X/\Gamma$$

whose kernel pair is the congruence Γ.

For any homomorphism $X \xrightarrow{\psi} Y$, we denote by $\operatorname{Ker}\psi$ the congruence on X determined by the kernel pair of ψ. We have a factorization of ψ

$$X \xrightarrow{p} X/\operatorname{Ker}\psi \xrightarrow{\bar{\psi}} \operatorname{Im}\psi \xrightarrow{q} Y$$

where $U_A(p)$, $U_A(\bar{\psi})$, $U_A(q)$ are respectively surjective, bijective, injective. It follows that $\bar{\psi}$ is an isomorphism.

<u>Proposition 4.5.</u> If

$$\begin{tikzcd} & X \arrow[dl, "\psi"'] \arrow[dr, "\theta"] & \\ Y \arrow[rr] & & Z \end{tikzcd}$$

is a commutative diagram in A^b, then $\operatorname{Ker}\psi \subseteq \operatorname{Ker}\theta$. Conversely, if Γ_1 and Γ_2 are congruences on X such that $\Gamma_1 \subseteq \Gamma_2$ then there is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ X/\Gamma_1 \arrow[rr] & & X/\Gamma_2 \end{tikzcd}$$

(ii) Colimits.

Now we turn to the construction of colimits. Let

$$X \overset{\phi}{\underset{\psi}{\rightrightarrows}} Y$$

be a diagram in A^b. Let Γ be the intersection of all congruences on Y whose underlying sets contain $\{(x\phi, x\psi) \mid x \in U_A(X)\}$. Let $Y \xrightarrow{\pi} Y/\Gamma$ be the projection.

<u>Proposition 4.6.</u> The diagram

$$X \overset{\phi}{\underset{\psi}{\rightrightarrows}} Y \xrightarrow{\pi} Y/\Gamma$$

is a coequalizer diagram.

<u>Proof.</u> Let $Y \xrightarrow{\epsilon} Z$ be a homomorphism such that $\phi\epsilon = \psi\epsilon$. Then

$$\{(x\phi, x\psi) \mid x \in U_A(X)\} \subseteq U_A(\mathrm{Ker}\,\epsilon)$$

and so $\Gamma \subseteq \mathrm{Ker}\,\epsilon$. If $y, y' \in U_A(Y)$ are such that $(y, y') \in U_A(\Gamma)$, then $y\epsilon = y'\epsilon$, so we may define a homomorphism $Y/\Gamma \xrightarrow{\mu} Z$ by $[y]\mu = y\epsilon$. Then $\pi\mu = \epsilon$, and since $U_A(\pi)$ is surjective, μ is unique.

Now we shall give a canonical method of presenting every A-model as a coequalizer of a pair of maps between free A-models. For any A-model X we write

$$D_0(X) = A_{U_A(X)}, \quad D_1(X) = A_{U_A F_A U_A(X)};$$

these are objects of A. Note that

$$I_A D_0(X) = F_A U_A(X), \quad I_A D_1(X) = (F_A U_A)^2(X).$$

Of course, $D_0 = j_A U_A$, $D_1 = j_A U_A F_A U_A$ are functors from A^b to A. Now we use the fact that $I_A : A \longrightarrow A^b$ is full and faithful. Let

$$(\delta_0)_X : D_1(X) \longrightarrow D_0(X)$$

be the unique map in A such that

$$I_A((\delta_0)_X) = \mathcal{E}_{F_A U_A(X)}$$

and let $(\delta_1)_X = j_A U_A(\mathcal{E}_X)$. Then $\delta_0, \delta_1 : D_1 \longrightarrow D_0$ are natural maps, and we have the diagram of functors $A^b \longrightarrow A$:

$$D : D_1 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} D_0 .$$

Thus $I_A D$ is a diagram of functors $A^b \longrightarrow A^b$. Let coeq denote the functor which to a diagram of the type above assigns its coequalizer. Then we have a functor

$$\text{coeq. } I_A D : A^b \longrightarrow A^b .$$

Theorem 4.7. The functor

$$\text{coeq. } I_A D : A^b \longrightarrow A^b$$

is naturally isomorphic to the identity functor.

Proof. We shall prove that

$$(F_A U_A)^2 X \underset{F_A U_A(\mathcal{E}_X)}{\overset{\mathcal{E}_{F_A U_A X}}{\rightrightarrows}} F_A U_A(X) \xrightarrow{\mathcal{E}_X} X$$

is a coequalizer diagram. Let us abbreviate it to

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \xrightarrow{\mathcal{E}} X$$

Let $X_0 \xrightarrow{\theta} Y$ be a homomorphism such that $d_0\theta = d_1\theta$. It is sufficient to prove that there exists a homomorphism $X \xrightarrow{h} Y$ such that $\varepsilon h = \theta$, as the surjectivity of $U_A(\varepsilon)$ will ensure that h is unique. Define $U_A(h)$ by

$$x \longmapsto (\delta_x^{U_A(X)})\theta .$$

Remember that $\{\delta_x^{U_A(X)}\}_{x \in U_A(X)}$ is a family of generators of X_0. Every element of $U_A(X_0)$ is of the form

$$y = \omega^\sigma \delta_{x_\sigma}^{U_A(X)}$$

where $\omega \in \Omega_S(A)$ and $\{x_\sigma\}_{\sigma \in S} \in X(A_s)$, for some set S. The homomorphisms d_0 and d_1 are given on the generators

$$\bar{y} = \delta_y^{U_A F_A U_A(X)}$$

of X_1, by the formulae

$$yd_0 = \delta_{\omega^\sigma x_\sigma}^{U_A(X)}$$

$$yd_1 = y = \omega^\sigma \delta_{x_\sigma}^{U_A(X)} .$$

We have $\omega^\sigma(x_\sigma h) = \omega^\sigma.(\delta_{x_\sigma}^{U_A(X)})\theta = yd_1\theta = yd_0\theta = \delta_{\omega^\sigma x_\sigma}^{U_A(X)}\theta = (\omega^\sigma x_\sigma)h$,

so that h is a homomorphism. It follows that $\varepsilon h = \theta$, so the proof is complete. We have thus represented every A-model as a coequalizer of homomorphisms between free A-models in a canonical way.

Now we use the diagram of functors D to construct coproducts in A^b. Let $\{X_i\}_{i \in I}$ be a family of A-models. For each $i \in I$ we have the diagram $D(X_i)$ in A. From the coproduct of these diagrams in A, say $\Delta = \coprod_{i \in I} D(X_i)$:

$$\Delta : \coprod_i \mathcal{D}_1(X_i) \underset{\coprod(\delta_1)_{X_i}}{\overset{\coprod(\delta_0)_{X_i}}{\rightrightarrows}} \coprod_i \mathcal{D}_0(X_i)$$

Let $Y = \text{coeq. } I_A \Delta$. The canonical maps of diagrams in A, $D(X_i) \xrightarrow{u_i} \Delta$, give rise to maps $X_i \xrightarrow{v_i} Y$ for each $i \in I$. We claim that these are the canonical maps to the coproduct of the family $\{X_i\}_{i \in I}$. To see this, consider a family of homomorphisms

$$\left\{X_i \xrightarrow{f_i} Z\right\}_{i \in I}$$

indexed by I.

We get a unique map of diagrams

$$\theta : \coprod_i D(X_i) \longrightarrow D(Z)$$

making the diagram of diagrams

$$\begin{tikzcd} D(X_j) \arrow[rr, "u_j"] \arrow[dr, "D(f_j)"'] & & \Delta \arrow[dl, "\theta"] \\ & D(Z) & \end{tikzcd}$$

commute. Applying coeq. $I_A(\text{—})$ we get commuting diagrams in A^b: -

$$\begin{tikzcd} X_j \arrow[rr, "v_j"] \arrow[dr, "f_j"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

The uniqueness of h follows from considering the commutative diagram

$$\begin{tikzcd} I_A \Delta_0 \arrow[r, "p"] & Y \arrow[dr, "h"] & \\ & & Z \\ I_A D_0(X_j) \arrow[uu, "I_A(u_j)_0"] \arrow[r, "\varepsilon_{X_j}"] & X_j \arrow[uu, "v_j"] \arrow[ur, "f_j"'] & \end{tikzcd}$$

where p is the projection to the coequalizer. Since ph is unique, and $U_A(p)$ is surjective, h is unique.

Hence we may write $Y = \coprod_{i \in I} X_i$. Thus, an algebraic category is cocomplete.

While we are on the subject of colimit properties of algebraic categories we mention the following results. Call a pre-ordered set <u>α-directed</u>, for a cardinal α, if every set of elements of cardinality less than α has an upper bound. An α-directed system in a category is a functor into that category from an α-directed preordered set. Call an A-model <u>α-generated</u> if it is generated by a set of elements of cardinality less than α.

<u>Theorem 4.8.</u> If A is an α-bounded theory, U_A preserves colimits over α-directed systems.

<u>Proof.</u> Let I be an α-directed preordered set, and $T : I \longrightarrow A^b$ a functor. Let $B = \varinjlim U_A T$. We construct an A-model X with $U_A(X) = B$ as follows: since A is α-bounded it is enough to define the action $X(\omega)$ for ω of arity less than α. Let $\omega \in \Omega_S(A)$, where S has cardinality less than α, and let $\{[b_\sigma, i_\sigma]\}_{\sigma \in S}$ be an S-indexed family of elements of B. Remember that these are of the form $[b, i]$ where $i \in I$, $b \in T(i)$ and $[b, i] = [b', i']$ if for some i'' such that $i \leqslant i''$ and $i' \leqslant i''$ the elements b and b' get taken to the same element in $T(i'')$. Define $\omega^\sigma[b_\sigma, i_\sigma]$ to be $[\omega^\sigma \bar{b}_\sigma, \bar{\iota}]$ where $\bar{\iota}$ is an upper bound of $\{i_\sigma\}_{\sigma \in S}$ and b_σ gets taken to $\bar{b}_\sigma$ under the image of T on $i_\sigma \leqslant \bar{\iota}$. This gives an A-model X, and the function $U_A T(i) \longrightarrow U_A(X)$ given by $x \longrightarrow [x, i]$ gives a coherent family of homomorphisms $T(i) \xrightarrow{u_i} X$. If $T(i) \xrightarrow{v_i} Y$ is any other coherent family of homomorphisms, define $X \xrightarrow{h} Y$ by $[x, i]h = xv_i$. Then $u_i h = v_i$, and clearly h is unique. Hence $X = \varinjlim T$. Thus $U_A(\varinjlim T) = \varinjlim U_A T$.

<u>Theorem 4.9.</u> If A is an α-bounded theory, every A-model is the colimit of its α-generated submodels.

Proof. It is enough to prove this result for free models, since every model is a quotient of a free model. The result now follows from the remark at the end of § 2 that for an α-bounded theory, every element of a free model may be represented by the application of an operation of arity less than α to a family of less than α generators.

Exercises 4.

1. Let A be a theory with a 3-ary operation θ satisfying

$$\theta(x, x, y) = y = \theta(y, x, x).$$

Show that a submodel of an A-model XxX is a congruence on X if and only if it contains the diagonal submodel, i.e. the image of the homomorphism

$$X \xrightarrow{\langle 1_X, 1_X \rangle} X \times X .$$

2. If A is an algebraic theory and f is a homomorphism of A-models, show that f is the coequalizer of its kernel pair if and only if $U_A(f)$ is surjective.

3. An epic map is called regular if it is a coequalizer of some pair of maps. Show that in algebraic categories, pullbacks of regular epics are regular epics.

4. Show that in an algebraic category, pullback along regular epics is an isomorphism reflecting functor.

§ 5. Maps of Theories

The notion of a map between theories is one which is new to the categorical approach. In the classical approach, the theory was given , and although its quotient theories were in effect studied, the notions of subtheory, or, in general, maps between theories, were simply not considered. Nevertheless, the concept is a natural one, and leads to some interesting questions.

Definition 5.1 If A and B are algebraic theories, a functor $f : A \longrightarrow B$ is a map of theories if it preserves coproducts and $f(A_1) = B_1$.

Such a map of theories $f : A \longrightarrow B$, induces for each S a function

$$\Omega_S(f) : \Omega_S(A) \longrightarrow \Omega_S(B)$$

and it is clear that these functions determine f uniquely. Furthermore, if A is α-bounded, f is determined by the $\Omega_S(f)$ for which the cardinality of S is less than α. Hence, if A is a bounded theory, there is only a set of maps of theories from A to B.

In this way we have a category Bth, whose objects are bounded theories, and whose maps are maps of theories.

Let us look at some examples of maps of theories:

a) (abelian groups) $\longrightarrow$ (rings)

b) (groups) $\longrightarrow$ (abelian groups)

c) (Lie-rings) $\longrightarrow$ (rings)

In example a) the addition for abelian groups is taken to the addition for rings. In b) group multiplication is taken to addition in abelian groups. In c) the

Lie-bracket is taken to the commutator.

If $A \xrightarrow{f} B$ is a map of theories so that for each S, $\Omega_S(f)$ is an inclusion map, we call A a <u>subtheory</u> of B, and f the inclusion. If for each S, $\Omega_S(f)$ is a surjection, we call B a <u>quotient theory</u> of A. In example b) above, we may say that (abelian groups) is a quotient theory of (groups).

For any theory A, the functor $j_A : \underline{S} \longrightarrow A$ is a map of theories. Indeed, up to natural isomorphism, it is the only map of theories from $\underline{S}$ to A. Thus $\underline{S}$ is an initial theory. Similarly $\mathbb{1}$ is a terminal theory.

If $A \xrightarrow{f} B$ is a map of theories and $Y : B^0 \longrightarrow \underline{S}$ is a B-model, then the composite

$$A^0 \xrightarrow{f} B^0 \xrightarrow{Y} \underline{S}$$

preserves products, and so is an A-model, which we denote by $f^b(Y)$. If $Y \xrightarrow{\theta} Y'$ is a homomorphism of B-models, the natural map $\theta . f$ is a homomorphism of A-models, which we denote by $f^b(\theta)$.

In this way we obtain a functor

$$f^b : B^b \longrightarrow A^b$$

which we call the "forgetful functor" or "pullback" along f.

It is an immediate consequence of the fact that $f(A_1) = B_1$ that the diagram

$$\begin{tikzcd} B^b \arrow[rr, "f^b"] \arrow[dr] & & A^b \arrow[dl] \\ & \underline{S} & \end{tikzcd}$$

commutes. Indeed, if we identify $\underline{S}^b$ with $\underline{S}$ we may identify U_A with j_A^b.

The associativity of functorial composition tells us that if we write

$$A \xrightarrow{f} B \xrightarrow{g} C \; = \; A \xrightarrow{fg} C$$

then

$$(fg)^b = f^b g^b .$$

For the examples given above, we have

a) $(\text{rings})^b \longrightarrow (\text{abelian groups})^b$: forget the multiplication structure.

b) $(\text{abelian groups})^b \longrightarrow (\text{groups})^b$: the inclusion functor.

c) $(\text{rings})^b \longrightarrow (\text{Lie-rings})^b$: treat a ring as a Lie-ring with commutator for the Lie-operation.

Note that the unique map of theories $A \xrightarrow{u} \mathbb{1}$ induces the functor $\mathbb{1} \xrightarrow{u^b} A^b$ which picks out the terminal object of A^b.

Now we prove an extension theorem, which will frequently prove useful.

Theorem 5.1. Let A be an algebraic theory, let C be a cocomplete category, and let

$$T : A \longrightarrow C$$

be a coproduct preserving functor. Then there is a unique colimit preserving functor

$$\tilde{T} : A^b \longrightarrow C$$

such that the diagram

$$\begin{tikzcd} A \arrow[r, "I_A"] \arrow[d, "T"'] & A^b \arrow[dl, dashed, "\tilde{T}"] \\ C & \end{tikzcd}$$

commutes.

Proof. For $X \in A^b$, define $\widetilde{T}(X)$ to be coeq. $TD(X)$, where D is the diagram of functors $D_1 \rightrightarrows D_0 : A^b \longrightarrow \bar{A}$ defined in § 4. This defines a functor $\widetilde{T} : A^b \longrightarrow C$. Next we wish to show that $\widetilde{T} . I_A = T$. For this purpose, recall that a contractible coequalizer diagram is a diagram of the form

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"'] & X_0 \arrow[l, shift left=4, "s_0"] \arrow[r, shift left, "d"] & X \arrow[l, shift left, "s"] \end{tikzcd}$$

such that the following four identities hold: -

a) $d_0 d = d_1 d$

b) $sd = 1_X$

c) $s_0 d_0 = 1_{X_0}$

d) $s_0 d_1 = ds$.

Lemma 5.2. With the same notation as above

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \xrightarrow{d} X$$

is a coequalizer diagram.

Proof. Let $X_0 \xrightarrow{\theta} Y$ be such that $d_0\theta = d_1\theta$. Let $h = s\theta$. Then

$$dh = ds\theta = s_0 d_1 \theta = s_0 d_0 \theta = \theta$$

and h is unique with this property because b) implies that d is epic.

Note that because contractible coequalizer diagrams are defined equationally, they are preserved by functors. Hence, any coequalizer diagram that is part of a contractible coequalizer diagram is preserved by functors.

Let $C \underset{U}{\overset{F}{\rightleftarrows}} D$ be functors with F left adjoint to U, with front and end adjunctions η and ε respectively. Then we have: -

<u>Lemma 5.3.</u> For any object S of C,

$$\begin{tikzcd}
(FU)^2F(S) \arrow[r, shift left, "\varepsilon FUF(S)"] \arrow[r, shift right, "FU\varepsilon F(S)"'] & FUF(S) \arrow[r, "\varepsilon F(S)"] \arrow[l, bend left=40, "FUF\eta S"] & F(S) \arrow[l, bend left=40, "F\eta S"]
\end{tikzcd}$$

is a contractible coequalizer diagram.

<u>Proof.</u> a) and d) follow from naturality; b) and c) follow from the properties of adjunctions.

Now we continue with the proof of theorem 5.1. We have to show that $\tilde{T}(F_A(S)) = T(A_S)$. We have a contractible coequalizer diagram

$$\begin{tikzcd}
(F_AU_A)^2F_A(S) \arrow[r, shift left, "\varepsilon F_AU_AF_A(S)"] \arrow[r, shift right, "F_AU_A\varepsilon F_A(S)"'] & F_AU_AF_A(S) \arrow[r, "\varepsilon F_A(S)"] \arrow[l, bend left=40, "F_AU_AF_A\eta S"] & F_A(S) \arrow[l, bend left=40, "F_A\eta S"]
\end{tikzcd}$$

in A^b. Since $I_A : A \longrightarrow A^b$ is full and faithful we have a contractible coequalizer diagram

$$\begin{tikzcd}
\mathcal{D}_1(F_A(S)) \arrow[r, shift left, "(\delta_0)_{F_A(S)}"] \arrow[r, shift right, "(\delta_1)_{F_A(S)}"'] & \mathcal{D}_0(F_A(S)) \arrow[r] \arrow[l, bend left=40] & A_S \arrow[l, bend left=40]
\end{tikzcd}$$

in A. Applying the functor T, we get the desired result. Since colimits commute with colimits, T is colimit preserving. Since every A-model is a colimit of free A-models, T is unique.

We call $\tilde{T} : A^b \longrightarrow C$, the <u>extension</u> of $T : A \longrightarrow C$.

Suppose we have a map of theories $A \xrightarrow{f} B$. Since $B \xrightarrow{I_B} B^b$ is coproduct preserving, so is $A \xrightarrow{f} B \xrightarrow{I_B} B^b$. Let

$$A^b \xrightarrow{f_*} B^b$$

be its extension.

<u>Theorem 5.4.</u> Let $A \xrightarrow{f} B$ be a map of theories. The functor

$$f_* : A^b \longrightarrow B^b$$

is left adjoint to $f^b : B^b \longrightarrow A^b$.

Proof. Let X be an A-model and Y a B-model. We have the following string of natural isomorphisms:

$$\mathrm{Hom}_{B^b}(f_*(X), Y) = \mathrm{Hom}_{B^b}(\text{coeq. } I_B f\, D(X), Y) \simeq$$

$$\simeq \text{eq. } \mathrm{Hom}_{B^b}(I_B f\, D(X), Y)$$

$$\simeq \text{eq. } Y(f\, D(X)) \qquad \text{by Yoneda lemma}$$

$$\simeq \text{eq. } \mathrm{Hom}_{A^b}(I_A D(X), Yf)$$

$$\simeq \mathrm{Hom}_{A^b}(\text{coeq. } I_A D(X), f^b(Y))$$

$$\simeq \mathrm{Hom}_{A^b}(X, f^b(Y)).$$

In the above, "eq" denotes equalizer of the pair of maps in the diagram denoted by the symbols which follow it.

This result is of great importance. It may be regarded as a relativization of the adjointness of U_A and F_A. Indeed, if we take the case $f = j_A : \underline{S} \longrightarrow A$ then, up to natural isomorphism, we have $j_A^b = U_A$, $(j_A)_* = F_A$.

Let us look at f_* for the examples given above:

a) $(\text{abelian groups})^b \longrightarrow (\text{rings})^b$ is the tensor algebra functor.

b) $(\text{groups})^b \longrightarrow (\text{abelian groups})^b$ is the functor

$$G \longrightarrow G/[G, G] .$$

c) $(\text{Lie-rings})^b \longrightarrow (\text{rings})^b$ is the universal enveloping ring functor.

The uniqueness of adjoints up to natural isomorphism ensures the coherent natural isomorphisms

$$(fg)_* \simeq g_* f_* .$$

If $A \xrightarrow{f} B$ is a map of theories and B is a bounded theory, say α-bounded, we may write down an explicit formula for f_*X for any A-model X as follows:

Consider the set

$$\coprod_T (\text{Hom}_B(B_S, B_T) \times X(A_T))$$

where the coproduct ranges over sets of cardinality less than α. On this set consider the equivalence relation generated by

$$(\beta . f(\omega), \xi) \quad (\beta, \omega . \xi)$$

where $B_S \xrightarrow{\beta} B_U$, $A_U \xrightarrow{\omega} A_T$ and $\xi \in X(A_T)$, for all β, ω, ξ, U and T. Denote the equivalence class containing (γ, ξ) by $\gamma \otimes_A \xi$.

Proposition 5.5. The elements of $f_*(X)(B_S)$ are in bijective correspondence with the equivalence classes $\gamma \otimes_A \xi$. The action of $\beta \in B$ corresponds to the function

$$\gamma \otimes_A \xi \longmapsto \beta\gamma \otimes_A \xi$$

We leave the verification as an exercise to the reader. This description of $f_*(X)$ only makes sense when B is bounded, even though $f_*(X)$ exists when B is unbounded. Note the similarlity of this construction with that for tensor products of modules.

If A is an algebraic theory, an A-theory is a pair (B, f) where B is an algebraic theory and $A \xrightarrow{f} B$ is a map of theories. A map of A-theories $(B, f) \longrightarrow (B', f')$ is simply a map of theories $B \xrightarrow{g} B'$ such that $g.f = f'$. For example, a $\mathbb{Z}$-theory is what one might call a linear theory.

Exercises 5

1. Show that a ring homomorphism $K \longrightarrow K'$ induces a map of theories $\mathrm{Mat}(K) \longrightarrow \mathrm{Mat}(K')$. Show that Mat is a functor

$$(\text{Rings})^b \longrightarrow \text{Bth}$$

and that it is full and faithful.

2. Let $K \xrightarrow{f} K'$ be a ring homomorphism. Interpret f^b and f_* and show that f^b has a right adjoint.

3. The identification of a monoid with a unary theory gives a functor

$$(\text{Monoids})^b \longrightarrow \text{Bth}.$$

Show that it is full and faithful and left adjoint to the functor

$$\Omega_1 : \text{Bth} \longrightarrow (\text{Monoids})^b .$$

4. Let Bth_0 be the full subcategory of bounded theories with no nullary operations. Show that the inclusion functor $\mathrm{Bth}_0 \hookrightarrow \mathrm{Bth}$ has a right adjoint, given by $A \longmapsto \bar{A}$.

5. Let $\mathrm{Aff\ Bth}_0$ be the full subcategory of Bth_0 of affine theories. Show that the inclusion functor $\mathrm{Aff\ Bth}_0 \hookrightarrow \mathrm{Bth}_0$ has a right adjoint given by $A \longmapsto \mathrm{Aff}(A)$.

6. Show that Conv (ex. 2.1.) is isomorphic to a subtheory of $\mathrm{Aff}(\mathbb{R})$, where $\mathbb{R}$ is the ring of real numbers.

§ 6. The tripleability of forgetful functors

No discussion of algebraic theories would be complete without mention of monads (or triples). Indeed, the whole subject may be rephrased in terms of them. We content ourselves here with a brief outline, since there are adequate texts elsewhere. We refer the reader particularly to chapter VI of Categories for the Working Mathematician [MacLane 3].

A monad (or triple) on a category C is given by a functor

$$T : C \longrightarrow C$$

and natural maps

$$\mu : T^2 \longrightarrow T \qquad \eta : 1_C \longrightarrow T$$

satisfying the conditions that the following diagrams commute:-

(Associativity)

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd}$$

(Unit)

$$\begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "1_T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "\eta T"'] \arrow[dl, "1_T"] \\ & T & \end{tikzcd}$$

A map of monads from $\underline{T} = (T, \mu, \eta)$ to $\underline{T}' = (T', \mu', \eta')$ is a natural map

$$\theta : T \longrightarrow T'$$

such that the following diagrams commute:-

$$\begin{tikzcd} T^2 \arrow[r, "\theta\theta"] \arrow[d, "\mu"'] & T'^2 \arrow[d, "\mu'"] \\ T \arrow[r, "\theta"] & T' \end{tikzcd} \qquad \begin{tikzcd} & 1_C \arrow[dl, "\eta"'] \arrow[dr, "\eta'"] & \\ T \arrow[rr, "\theta"] & & T' \end{tikzcd}$$

<u>Proposition 6.1.</u> Let $C \underset{G}{\overset{F}{\rightleftarrows}} D$ be a pair of adjoint functors, with F left adjoint to G, with unit $\eta : 1_C \longrightarrow GF$ and counit $\varepsilon : FG \longrightarrow 1_D$. Then $(GF, G\varepsilon F, \eta)$ is a monad on C.

If $\underline{T} = (T, \mu, \eta)$ is a monad on C, a <u>$\underline{T}$-algebra</u> is a pair (X, ξ) where X is an object of C and

$$\xi : T(X) \longrightarrow X$$

is a map such that the diagrams

$$\begin{tikzcd} T^2(X) \arrow[r, "T(\xi)"] \arrow[d, "\mu_X"'] & T(X) \arrow[d, "\xi"] \\ T(X) \arrow[r, "\xi"] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] & T(X) \arrow[d, "\xi"] \\ & X \end{tikzcd}$$

commute.

A map of $\underline{T}$- algebras $(X, \xi) \longrightarrow (X', \xi')$ is given by a map $X \xrightarrow{\alpha} X'$ in C such that the diagram

$$\begin{tikzcd} T(X) \arrow[r, "T(\alpha)"] \arrow[d, "\xi"'] & T(X') \arrow[d, "\xi'"] \\ X \arrow[r, "\alpha"] & X' \end{tikzcd}$$

commutes.

We obtain a category $C^{\underline{T}}$ of $\underline{T}$-algebras and maps of $\underline{T}$-algebras.

We have functors

$$U^{\underline{T}} : C^{\underline{T}} \longrightarrow C \qquad\qquad (X, \xi) \longmapsto X$$

and $$F^{\underline{T}} : C \longrightarrow C^{\underline{T}} \qquad\qquad A \longrightarrow (T(A), \mu_A).$$

<u>Proposition 6.2.</u> The functor $F^{\underline{T}}$ is left adjoint to $U^{\underline{T}}$, and the adjoint pair $(F^{\underline{T}}, U^{\underline{T}})$ give rise to the monad $\underline{T}$ on C.

We call a $\underline{T}$-algebra which is isomorphic to one of the form $F^{\underline{T}}(A)$ for some A in C, a <u>free</u> $\underline{T}$-algebra. Let $C_{\underline{T}}$ be the full subcategory of $C^{\underline{T}}$ of free $\underline{T}$-algebras. It is clear that the adjoint pair

$$C \underset{U^{\underline{T}}}{\overset{F^{\underline{T}}}{\rightleftarrows}} C^{\underline{T}}$$

restrict to give an adjoint pair

$$C \underset{U_{\underline{T}}}{\overset{F_{\underline{T}}}{\rightleftarrows}} C_{\underline{T}}$$

which again gives rise to the monad $\underline{T}$ on C. Notice that every object of $C_{\underline{T}}$ is isomorphic to an object in the image of $F_{\underline{T}}$. The category $C^{\underline{T}}$ (resp. $C_{\underline{T}}$) is called the Eilenberg-Moore (resp. Kleisli) category of the monad $\underline{T}$. There were discovered independently, and both give a converse to proposition 6.1. Every monad arises from an adjoint pair. They each play a universal role, according to the following proposition:

<u>Proposition 6.3.</u> Let $C \underset{G}{\overset{F}{\rightleftarrows}} D$ be a pair of adjoint functors with F left adjoint to G, inducing the monad $\underline{T}$ on C. Then there are unique functors

$$J : C_{\underline{T}} \longrightarrow D , \qquad \Phi : D \longrightarrow C^{\underline{T}}$$

such that the diagrams

$$\begin{tikzcd}
C_{\underline{T}} \arrow[r, "J"] \arrow[dr, "U_{\underline{T}}"'] & D \arrow[r, "\Phi"] \arrow[d, "G"] & C^{\underline{T}} \arrow[dl, "U^{\underline{T}}"] \\
& C &
\end{tikzcd}$$

$$\begin{tikzcd}
C_{\underline{T}} \arrow[r, "J"] & D \arrow[r] & C^{\underline{T}} \\
& C \arrow[ul, "F_{\underline{T}}"] \arrow[u, "F"] \arrow[ur, "F^{\underline{T}}"'] &
\end{tikzcd}$$

commute.

It follows that the composite $C_{\underline{T}} \xrightarrow{J} D \xrightarrow{\Phi} C^{\underline{T}}$ is the full and faithful embedding $C_{\underline{T}} \longrightarrow C^{\underline{T}}$.

The functor $G : D \longrightarrow C$ is called <u>strongly tripleable</u> if the functor $\Phi : D \longrightarrow C^{\underline{T}}$ is an isomorphism of categories.

<u>Proposition 6.4.</u> If $\underline{T}$ is a monad on $\underline{S}$, then $\underline{S}_{\underline{T}}$ is an algebraic theory.

Proof : Since $F_{\underline{T}}$ has a right adjoint it preserves coproducts. Every object of $\underline{S}_{\underline{T}}$ is isomorphic to one of the form $F_{\underline{T}}(S)$ for some S, and hence to $\coprod_{S} F_{\underline{T}}(1)$.

It is easy to see that a map of monads $\underline{T} \longrightarrow \underline{T}'$ induces a map of theories $\underline{S}_{\underline{T}} \longrightarrow \underline{S}_{\underline{T}'}$.

<u>Proposition 6.5.</u> The functor (between illegitimate categories) from monads on $\underline{S}$ to algebraic theories, given by $\underline{T} \longrightarrow \underline{S}_{\underline{T}}$, is an equivalence.

The inverse equivalence is that which assigns to a theory A the monad induced by the adjoint pair (F_A, U_A). Let us call this monad $\underline{T}(A)$. These results tell us that we may identify $\underline{S}_{\underline{T}(A)}$ with A, $\underline{S}^{\underline{T}(A)}$ with A^b, the inclusion

$$\underline{S}_{\underline{T}(A)} \longrightarrow \underline{S}^{\underline{T}(A)}$$

with

$$A \xrightarrow{I_A} A^b$$

and the adjoint pair $(F^{\underline{T}}, U^{\underline{T}})$ with (F_A, U_A). In particular, the forgetful functor U_A is strongly tripleable. The proof given in the text cited above is easily extended to proving the following relative version:

<u>Proposition 6.6.</u> Let $A \xrightarrow{f} B$ be a map of theories. Then the functor

$$f^b : B^b \longrightarrow A^b$$

is strongly tripleable.

This tells us that we may regard B-models as A-models with a $\underline{T}_f$-structure, where $\underline{T}_f$ is the monad on A^b induced by the adjoint pair (f_*, f^b). This alternative way of looking at things can be very convenient. We may prove that every monad on A^b arises in this way (up to isomorphism of monads).

Suppose that $T \xrightarrow{\theta} T'$ determines a map of monads $\underline{T} \longrightarrow \underline{T}'$ on a category C. We have a functor

$$\theta^b : C^{\underline{T}'} \longrightarrow C^{\underline{T}}$$

given by

$$(X, \xi) \longmapsto (X, \xi\theta_X)$$

If C has coequalizers, this functor has a left adjoint

$$\theta_* : C^{\underline{T}} \longrightarrow C^{\underline{T}'}$$

given by the coequalizer diagram

$$T'T(X) \underset{\mu'_X\, T'\theta}{\overset{T'\xi}{\rightrightarrows}} T'(X) \longrightarrow \theta_*(X,\xi)$$

If $A \xrightarrow{f} B$ is a map of theories, the identification of A^b with $\underline{S}^{\underline{T}(A)}$ identifies $T(f)^b$ with f^b and $T(f)_*$ with f_*. It is useful to have the two pictures - theories and monads. Each has its own advantages. We could have described the correspondence between the two pictures with much greater pedantry, but for simplicity we shall simply identify A^b with $\underline{S}^{\underline{T}(A)}$ (and $\underline{S}^{\underline{T}}$ with $(\underline{S}_{\underline{T}})^b$ for any arbitrary monad $\underline{T}$ on $\underline{S}$).

If α is a cardinal, we say that a monad $\underline{T}$ on $\underline{S}$ is <u>α-bounded</u> if for any set S and element $x \in T(S)$ there is a set V of cardinality less than α and a function $V \xrightarrow{f} S$ such that x is in the image of $T(f)$. With this definition, a monad is α-bounded if and only if its associated algebraic theory is α-bounded.

§ 7. Semantics

We have seen how a theory A determines a functor

$$A^b \xrightarrow{U_A} \underline{S}$$

and a map of theories $A \xrightarrow{f} B$ determines a commutative diagram

$$\begin{tikzcd} A^b \arrow[dr] & & B^b \arrow[ll, "f^b"'] \arrow[dl] \\ & \underline{S} & \end{tikzcd}$$

This gives us a functor, which we call semantics, from the illegitmate category of theories to the illegitimate category of categories over $\underline{S}$.

The functor U_A determines the theory A, because if S is a set

$$\mathrm{Nat}(\prod_S U_A, U_A) \simeq \mathrm{Nat}(F_A, F_A \coprod_S) \simeq \Omega_S(A).$$

Similarly f^b determines f. Composition with f^b gives a function

$$\mathrm{Nat}(\prod_S U_A, U_A) \longrightarrow \mathrm{Nat}(\prod_S U_B, U_B)$$

which corresponds to $\Omega_S(f)$.

We call a functor $\underline{C} \xrightarrow{U} \underline{S}$ tractable if for any set S, the class of natural maps from $\prod_S U$ to U is a set. For any tractable functor $\underline{C} \xrightarrow{U} \underline{S}$ we define an algebraic theory

$$\mathrm{Str}(U)$$

by taking the dual of the full subcategory of $\underline{S}^{\underline{C}}$ (an illegitimate category) given by the functors $\prod_S U$, for S a set. The tractability of U gives us that Str(U) is a legitimate category. It is an algebraic theory with

$$\Omega_S(\mathrm{Str}(U)) = \mathrm{Nat}(\prod_S U,\ U).$$

If

$$\begin{tikzcd} \underline{C}' \arrow[rr, "F"] \arrow[dr, "U'"'] & & \underline{C} \arrow[dl, "U"] \\ & \underline{S} & \end{tikzcd}$$

is a commutative diagram, where U and U' are tractable, we obtain a map of theories

$$\mathrm{Str}(F) : \mathrm{Str}(U) \longrightarrow \mathrm{Str}(U')$$

by taking

$$\Omega_S(\mathrm{Str}(F)) : \mathrm{Nat}(\prod_S U, U) \longrightarrow \mathrm{Nat}(\prod_S U',\ U')$$

to be the function defined by composing with F.

In this way we obtain a functor, called algebraic structure, from the illegitimate category of tractable functors into $\underline{S}$ to the illegitimate category of algebraic theories.

Since $\mathrm{Str}(U_A) \simeq A$, and $\mathrm{Str}(f^b) \simeq f$, algebraic structure is left inverse to semantics. These functors are also adjoint. The other adjunction is given by a functor

$$\begin{tikzcd} \underline{C} \arrow[rr, "\varepsilon"] \arrow[dr, "U"'] & & \mathrm{Str}(U)^b \arrow[dl, "U_{\mathrm{Str}(U)}"] \\ & \underline{S} & \end{tikzcd}$$

which takes an object X in $\underline{C}$ to the Str(U)-model whose underlying set is U(X) and for which the action of a map $\alpha \in \mathrm{Str}(U)$ is given by evaluation of the natural map α at X.

<u>Corollary 7.1.</u> Let A and B be algebraic theories and let $F : B^b \longrightarrow A^b$ be a functor such that

$$\begin{tikzcd} B^b \arrow[rr, "F"] \arrow[dr, "U_B"'] & & A^b \arrow[dl, "U_A"] \\ & \underline{S} & \end{tikzcd}$$

commutes.

Then there is a unique map of theories $A \xrightarrow{f} B$ such that $F = f^b$.

Proof. Take $f = \mathrm{Str}(F)$.

Suppose that $\underline{C}$ is a category with coproducts. Then for any $X \in \underline{C}$, the functor

$$U = \mathrm{Hom}_{\underline{C}}(X, -) : \underline{C} \longrightarrow \underline{S}$$

is tractable, because

$$\mathrm{Nat}(\prod_S U, U) \simeq \mathrm{Hom}_{\underline{C}}(X, \coprod_S X).$$

$\mathrm{Str}(U)$ is clearly isomorphic to the full subcategory of $\underline{C}$ consisting of all coproducts of X. We obtain a diagram

$$\begin{tikzcd} & \mathrm{Str}(U) \arrow[dl, "i"'] \arrow[dr, "I_{\mathrm{Str}(U)}"] & \\ \underline{C} \arrow[rr, "\varepsilon"] \arrow[dr, "U"'] & & \mathrm{Str}(U)^b \arrow[dl, "U_{\mathrm{Str}(U)}"] \\ & \underline{S} & \end{tikzcd}$$

which commutes, where i is full and faithful.

Suppose that $\underline{C}$ has finite limits. We call an object X in C a <u>regular projective generator</u> if it satisfies the following condition : for any map f in C, f is a coequalizer of some pair of maps if and only if $\mathrm{Hom}_C(X, f)$ is a surjective function.

In [Lawvere, 1] Lawvere proves in the finitary case that

$$\mathcal{E} : C \longrightarrow Str(U)^b$$

is full, faithful and has a left adjoint if X is a regular projective generator. Further, $\mathcal{E}$ is an equivalence if, in addition, in C equivalence relations are kernel pairs.

If A is an algebraic theory, the object $F_A(1)$ is a regular projective generator in A^b. In general it is not the only one, and the others will determine equivalences $A^b \simeq B^b$ for some other algebraic theories B. When this happens we say that A and B are Morita equivalent.

We refer the reader to [Lawvere, 2] for a particularly interesting discussion of algebraic theories Str(U) for certain functors U.

Exercises 7.

1. Let U be the forgetful functor from fields and field extensions to $\underline{S}$. Show that $\mathrm{Str}(U)$ contains (commutative rings) and a unary operation θ satisfying:

$$\theta(1) = 1 \ , \qquad \theta(x.y) = \theta(x).\theta(y)$$
$$x^2.\theta(x) = x \ , \qquad \theta(\theta(x)) = x.$$

2. Let A be a theory, and X an A-model. Let (X, A^b) be the category whose objects are maps $X \longrightarrow Y$ in A^b and whose maps are commutative diagrams

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ Y \arrow[rr] & & Y' \end{tikzcd}$$

Show that with $(X \longrightarrow Y) \longrightarrow U_A(Y)$ as forgetful functor, the category (X, A^b) is algebraic.

3. If the algebraic theory of exercise 2 above is denoted by A_X, show that $\Omega_\emptyset(A_X) \simeq U_A(X)$. With the notation of § 3 show that $A \simeq \overline{A}_{F_A(\emptyset)}$ where $F_A(\emptyset)$ is interpreted as an $\overline{A}$-model via the inclusion functor $i^b : A^b \longrightarrow \overline{A}^b$ for $i : \overline{A} \longrightarrow A$.

4. Let J be the inclusion functor of the category of finite sets and functors into $\underline{S}$. Show that $\mathrm{Nat}(\prod_S J, J)$ is in bijective correspondence with the set of ultrafilters on S.

§8. Bimodels

We have seen that if B is an algebraic theory, the category B^b of B-models is cocomplete. If A is another algebraic theory, it makes sense to talk of A-models in $(B^b)^0$, or, to put it another way, of co-A-models in B^b. Such a gadget we call an <u>(A,B)-bimodel.</u>

More formally, an (A,B)-bimodel is a coproduct preserving functor $A \longrightarrow B^b$. A homomorphism of (A,B)-bimodels is to be a natural map between such functors. We shall denote the category of (A,B)-bimodels and homomorphisms of (A,B)-bimodels by $[A,B]$.

If $A \xrightarrow{X} B^b$ is an (A,B)-bimodel, we call $X(A_1)$ the <u>underlying</u> B-model of X, and we have an evident forgetful functor

$$U_{[A,B]}: \ [A,B] \longrightarrow B^b \ : \qquad X \longmapsto X(A_1).$$

We may identify $X(A_S)$ with $\coprod_S X(A_1)$ and we shall do this from hereon without comment. If $w \in \Omega_S(A)$ is an S-ary operation of A, we have a homomorphism of B-models

$$X(\omega) \ : X(A_1) \longrightarrow \coprod_S X(A_1)$$

which we call the <u>coaction</u> of ω on X. Clearly, X is determined by the underlying B-model $X(A_1)$ and by the coactions $X(\omega)$, so that $U_{[A,B]}$ is a faithful functor.

One of the most popular examples of the concept of bimodels is afforded by Hopf algebras. If B = (commutative rings) then coproduct in B^b is given by $\otimes_{\mathbb{Z}}$, so that a (Gp,B)-bimodel is given by a commutative ring R, together with Gp-costructure, i.e. a comultiplication $R \longrightarrow R \otimes_{\mathbb{Z}} R$, a co-unit $R \longrightarrow \mathbb{Z}$, a coinverse $R \longrightarrow R$ and so on, satisfying appropriate axioms.

Another example, of fundamental importance, is given by (A,B)-bimodels when A and B are annular theories. In that case, an (A,B)-bimodel is simply an (A,B)-bimodule, i.e. an abelian group which has a left B-module and right A-module structure, with the left and right actions commuting with each other. An (A,B)-bimodule homomorphism is simply a homomorphism of bimodules.

Let X be an (A,B)-bimodel and Y be a B-model. Consider the composite functor

$$A^0 \xrightarrow{X} (B^b)^0 \xrightarrow{\mathrm{Hom}_{B^b}(\cdot, Y)} \underline{S} .$$

Both factors preserve products, so the composite is an A-model, which we denote by

$$\mathrm{Hom}_B(X, Y) .$$

This construction is clearly functorial, so we have a functor

$$\mathrm{Hom}_B(-,-) : [A, B]^0 \times B^b \longrightarrow A^b .$$

Note that $U_A(\mathrm{Hom}_B(X,Y)) = \mathrm{Hom}_{B^b}(U_{[A,B]}(X),(Y)$, i.e. the underlying set of the A-model $\mathrm{Hom}_B(X,Y)$ is the set of homomorphisms of B-models from the underlying B-model of X to Y.

We call a functor $B^b \longrightarrow A^b$ naturally isomorphic to one of the form $\mathrm{Hom}_B(X,-)$ for some (A,B)-bimodel X, a <u>representable functor.</u>

<u>Proposition 8.1</u> An (A,B)-bimodel X is determined by the representable

functor $\mathrm{Hom}_B(X,-)$ uniquely up to isomorphism. The (A,B)-bimodels X for which $U_{[A,B]}(X) = Y$ are in bijective correspondence with the liftings of the functor $\mathrm{Hom}_{B^b}(Y,-)$ to functors $T : B^b \longrightarrow A^b$ such that $U_A . T = \mathrm{Hom}_{B^b}(Y,-)$.

Proof. This is a straight application of the Yoneda lemma. If $U_{[A,B]}(X) = Y$, then $\mathrm{Hom}_B(X,-)$ plays the role of T, uniquely since U_A is faithful. Conversely, given the lifting T, for each map $A_S \xrightarrow{\alpha} A_W$ of A, we have an action of α

$$\prod_W \mathrm{Hom}_{B^b}(Y,-) \longrightarrow \prod_S \mathrm{Hom}_{B^b}(Y,-)$$

and so a map $\coprod_S Y \longrightarrow \coprod_W Y$, which defines an A-costructure on Y, giving us an (A,B)-bimodel with Y for underlying B-model.

This proposition allows us to determine a bimodel by means of the representable functor associated to it. For example, a cogroup structure on $\mathbf{Z}[t,t^{-1}]$ is determined by the representable functor from (commutative rings)^b to (groups)^b given by "invertible elements of (-)".
If $X : A \longrightarrow B^b$ is an (A,B)-bimodel, then by theorem 5.1, there exists a unique functor

$$\tilde{X} \;:\; A^b \longrightarrow B^b$$

which preserves colimits, such that $X = \tilde{X}.I_A$. We denote this functor by

$$X \otimes_A (-) \;:\; A^b \longrightarrow B^b \;.$$

According to §5, if Z is an A-model, then

$$X \otimes_A Z = \operatorname{coeq} X\, D(Z)\,.$$

It follows that we have a functor

$$(-) \otimes_A (-) \;:\; [A,B] \times A^b \longrightarrow B^b\,.$$

<u>Theorem 8.2</u> For any (A,B)-bimodel X, the functor $X \otimes_A (-)$ is left adjoint to $\operatorname{Hom}_B(X,-)$.

Proof: Let Y be a B-model, Z an A-model. We have the following sequence of natural bijections:

$$\operatorname{Hom}_{B^b}(X \otimes_A Z, Y) \simeq$$

$$\operatorname{Hom}_{B^b}(\operatorname{coeq} XD(Z), Y) \simeq$$

$$\text{eq. } \operatorname{Hom}_{B^b}(XD(Z), Y) \simeq$$

$$\text{eq. } \operatorname{Hom}_{A^b}(I_A . D(Z), \operatorname{Hom}_B(X,Y)) \simeq$$

$$\operatorname{Hom}_{A^b}(\operatorname{coeq.} I_A\, D(Z), \operatorname{Hom}_B(X,Y)) \simeq$$

$$\operatorname{Hom}_{A^b}(Z, \operatorname{Hom}_B(X,Y)).$$

Suppose now that A, B, C are algebraic theories, that L is an (A,B)-bimodel and M is a (B,C)-bimodel. Consider the commutative diagram

$$\begin{tikzcd}
A \arrow[d, "I_A"'] \arrow[dr, "L"] & B \arrow[d, "I_B"'] \arrow[dr, "M"] & \\
A^b \arrow[r, "L \otimes_A (-)"'] & B^b \arrow[r, "M \otimes_B (-)"'] & C^b
\end{tikzcd}$$

Beacuse $M \otimes_B(-)$ has a right adjoint it preserves coproducts, so the composite $M \otimes_B(-) \circ L$ is an (A, C)-bimodel, which we naturally denote by $M \otimes_B L$. Inspection of the diagram shows that

$$(M \otimes_B L) \otimes_A (-) \simeq M \otimes_B (L \otimes_A (-)) \quad .$$

From the uniqueness of adjoints it follows that

$$\mathrm{Hom}_C(M \otimes_B L, -) \simeq \mathrm{Hom}_B(L, \mathrm{Hom}_C(M, -))$$

Since composition of functors is associative, the bifunctor

$$(-) \otimes_B (-) : [B, C] \times [A, B] \longrightarrow [A, C]$$

is coherently associative. The (A, A)-bimodel

$$I_A : A \longrightarrow A^b$$

acts like a 2-sided unit for $\otimes_A$. Its underlying A-model is $F_A(1)$. When A is annular, I_A is simply the ring A itself considered as a bimodule of itself. It is common practice to abuse notation by using the same symbol for A and I_A, and we shall sometimes do this, so that the notation $X \otimes_A A \simeq X \simeq B \otimes_B X$ makes sense if X is an (A, B)-bimodel.

<u>Proposition 8.3</u> If X, X' are (A, B)-bimodels, every natural map $X \otimes_A (-) \xrightarrow{\lambda} X' \otimes_A (-)$ is of the form $\xi \otimes_A (-)$ for a unique homomorphism of (A, B)-bimodels $X \xrightarrow{\xi} X'$.

Proof. Define ξ to be $\lambda . I_A$.

<u>Corollary 8.4.</u> Every natural map $\mathrm{Hom}_B(X', -) \longrightarrow \mathrm{Hom}_B(X, -)$ is of the form $\mathrm{Hom}_B(\xi, -)$ for a unique homomorphism of (A, B)-bimodels $X \xrightarrow{\xi} X'$.

<u>Theorem 8.5</u> A functor $G : A^b \longrightarrow B^b$ is of the form $X \otimes_A (-)$ for some (A,B)-bimodel X if and only if it has a right adjoint.

Proof: One way round is clear, because $\mathrm{Hom}_B(X,-)$ is right adjoint to $X \otimes_A (-)$. For the other way, suppose that G has a right adjoint. Then it preserves coproducts, so

$$A \xrightarrow{I_A} A^b \xrightarrow{G} B^b$$

is an (A,B)-bimodel X. By the uniqueness of the lifting theorem 5.1, it follows that $G \simeq X \otimes_A (-)$.

<u>Corollary 8.6</u> A functor $F : B^b \longrightarrow A^b$ is representable if and only if it has a left adjoint.

If $A \xrightarrow{f} B$ is a map of algebraic theories, then

$$A \xrightarrow{f} B \xrightarrow{I_B} B^b$$

is an (A,B)-bimodel which we denote by Bf. Since the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "I_A"'] & B \arrow[d, "I_B"] \\ A^b \arrow[r, "f_*"] & B \end{tikzcd}$$

commutes, it follows that

$$Bf \otimes_A (-) = f_*$$
$$\mathrm{Hom}_B(Bf, -) = f^b .$$

Since a functor $B^b \longrightarrow A^b$ is of the form f^b for some map of theories f if and only if the diagram

$$\begin{tikzcd} \mathcal{B}^b \arrow[rr] \arrow[dr, "U_B"'] & & A^b \arrow[dl, "U_A"] \\ & \underline{S} & \end{tikzcd}$$

commutes, it follows

that an (A,B)-bimodel X is of the form Bf if and only if $U_{[A,B]}(X) \simeq F_B(1)$.

It is convenient to think of (A,B)-bimodels as generalized maps from A to B. Maps of theories in the strict sense are those which satisfy the condition alone. Composition of maps is given by using $\otimes$.

Suppose that $\{X_\nu\}$ is a diagram of (A,B)-bimodels. We may define an (A,B)-bimodel $\varinjlim_\nu X_\nu$ by the formulae

$$\varinjlim_\nu X_\nu\,(\alpha) = \varinjlim_\nu (X_\nu(\alpha)) \qquad \alpha \in A$$

because colimits commute with coproducts. We note the following formulae

$$\varinjlim_\nu X_\nu \otimes_A (-) \simeq \varinjlim_\nu (X_\nu \otimes_A (-))$$

$$\mathrm{Hom}_B\left(\varinjlim_\nu X_\nu, -\right) \simeq \varprojlim_\nu \mathrm{Hom}_B(X_\nu, -).$$

In particular, for any set S and theory A, we have the (A,A)-bimodel $\coprod_S I_A$ which we call the <u>free</u> bimodel on the set S. It represents the functor $\prod_S(-)$: $A^b \longrightarrow A^b$. A theorem of Kan, asserts that every (Gp,Gp)-bimodel is free.

Exercises 8

1. If A,B are unary (resp. annular) theories, show that $[A,B]$ with forgetful functor

$$[A,B] \xrightarrow{U_{[A,B]}} B^b \xrightarrow{U_B} \underline{S}$$

is algebraic, and is the category of models of a unary (resp. annular) theory.

2. If $R[[t]]$ denotes the ring of power series in the indeterminate t over the ring R, show that the functor $R \longrightarrow R[[t]]$ is representable. Is the functor $R \longrightarrow R[t]$ representable?

3. Let A be the theory of commutative $\mathbb{Z}_p$-algebras, where p is a prime number. Show that the functor which takes a $\mathbb{Z}_p$-algebra to the group of invertible elements of order p is representable.

4. If A and B are theories, show that $[A,B]$ has colimits, given by $\left(\varinjlim_{\nu} X_\nu\right)(\alpha) = \varinjlim_{\nu} (X_\nu(\alpha))$.

§9. Algebras over theories

Let A be an algebraic theory. By analogy with the annular case, we define an A-algebra to be a triple (X, m, e) where X is an (A,A)-bimodel, and

$$m : X \otimes_A X \longrightarrow X \qquad e : I_A \longrightarrow X$$

are homomorphisms of (A,A)-bimodels making the diagrams

$$\begin{tikzcd} X \otimes_A X \otimes_A X \arrow[r, "m \otimes_A 1_X"] \arrow[d, "1_X \otimes_A m"'] & X \otimes_A X \arrow[d, "m"] \\ X \otimes_A X \arrow[r, "m"] & X \end{tikzcd}$$

$$\begin{tikzcd} I_A \otimes_A X \arrow[r, "e \otimes_A 1_X"] \arrow[dr, "\sim"'] & X \otimes_A X \arrow[d, "m"] & X \otimes_A I_A \arrow[l, "1_X \otimes_A e"'] \arrow[dl, "\sim"] \\ & X & \end{tikzcd}$$

commute.

A <u>homomorphism of A-algebras</u> $(X, m, e) \longrightarrow (X', m', e')$ is given by a homomorphism of (A,A)-bimodules

$$f : X \longrightarrow X'$$

such that the diagrams

$$\begin{tikzcd} X \otimes_A X \arrow[r, "f \otimes_A f"] \arrow[d, "m"'] & X' \otimes_A X' \arrow[d, "m'"] \\ X \arrow[r, "f"] & X' \end{tikzcd} \qquad \begin{tikzcd} & I_A \arrow[dl, "e"'] \arrow[dr, "e'"] & \\ X \arrow[rr, "f"] & & X' \end{tikzcd}$$

commute.

In this way we obtain a category A-alg of A-algebras and homomorphisms of A-algebras. It should be clear from proposition 8.3 that an A-algebra structure on an (A, A)-bimodel X is simply the same thing as a monad structure on $X \otimes_A (-)$ and that a homomorphism of A-algebras corresponds to a map of monads.

We have a forgetful functor

$$\text{A-alg} \longrightarrow [A, A]$$

which we shall not bother to name. We shall also adopt the convention of abbreviating the symbol (X, m, e) to simply X.

If X is an A-algebra, an <u>X-module</u> is to be an algebra of the associated monad $X \otimes_A (-)$. That is to say, an X-module is an A-model M together with a map, the structure map,

$$X \otimes_A M \xrightarrow{\mu} M$$

satisfying the usual axioms. We could, of course, equally well describe an X-module as a coalgebra of the comonad $\mathrm{Hom}_A(X, -)$, with costructure map

$$M \xrightarrow{\tilde{\mu}} \mathrm{Hom}_A\ (X, M)$$

adjoint to the structure map. A homomorphism of X-modules is a map of $X \otimes_A (-)$-algebras, or, equivalently, a map of $\mathrm{Hom}_A(X, -)$-coalgebras.

The inspiration for our terminology is taken, as usual, from the annular case. If R is a ring, let us stretch the usual terminology by defining an R-algebra to be an (R, R)-bimodule S together with maps of bimodules

$$R \xrightarrow{e} S \quad , \quad S \otimes_R S \xrightarrow{m} S$$

satisfying the usual axioms. Then S becomes a ring, and e becomes a ring homomorphism. In fact all ring homomorphism with domain R are obtained in this way.

Correspondingly, we shall construct a full and faithful functor from A-alg to the category of A-theories. However, for theories all does not work so smoothly as for rings, because not every A-theory arises from an A-algebra. We shall show that an A-theory $A \xrightarrow{f} B$ arises from an A-algebra if and only if $f^b : B^b \longrightarrow A^b$ has a right adjoint as well as a left adjoint.

We shall anticipate the construction of the functor from A-algebras to A-theories by a few deliberate abuses of notation. First, if

$$X = (X, m, e)$$

is an A-algebra, we shall denote by X^b the category of X-modules and homomorphism of X-modules , and by

$$e^b : X^b \longrightarrow A^b$$

the forgetful functor $(M,\mu) \longrightarrow M$. This functor has a left adjoint

$$e_* : A^b \longrightarrow X^b \qquad : \qquad N \longrightarrow (X \otimes_A N,\ m \otimes_A 1_N)$$

which takes an A-model N to the free $X \otimes_A (-)$-algebra on N, and also a right adjoint

$$e_+ : A^b \longrightarrow X^b \qquad : \qquad N \longrightarrow (\mathrm{Hom}_A(X,N),\ \tilde{m}_N)$$

which takes an A-model N to the cofree $\mathrm{Hom}_A(X, -)$-coalgebra on N. Here, $\tilde{m}$ is the comultiplication of the comonad $\mathrm{Hom}_A(X, -)$ adjoint to m.

Suppose that $X_1 \xrightarrow{f} X_2$ is a map of A-algebras. If (M,μ) is an X_2-module, then $(M, \mu . f \otimes_A 1_M)$ is an X_1-module, with structure map

$$X_1 \otimes_A M \xrightarrow{f \otimes_A 1_M} X_2 \otimes_A M \xrightarrow{\mu} M$$

so we have a functor, pullback along f,

$$f^b : X_2^b \longrightarrow X_1^b .$$

It is clear that the diagram

$$\begin{tikzcd} X_2^b \arrow[rr, "f^b"] \arrow[dr, "\varepsilon_2^b"'] & & X_1^b \arrow[dl, "\varepsilon_1^b"] \\ & A^b & \end{tikzcd} \tag{9.1}$$

commutes.

The functor f^b has a left adjoint

$$f_* : X_1^b \longrightarrow X_2^b$$

given as follows: let (M,μ) be an X_1-module; form the coequalizer in A^b of the maps

$$\begin{tikzcd} X_2 \otimes_A X_1 \otimes_A M \arrow[rr, "1_{X_2} \otimes_A \mu"] \arrow[dr, "1_{X_2} \otimes_A f \otimes 1_M"'] & & X_2 \otimes_A M \\ & X_2 \otimes_A X_2 \otimes_A M \arrow[ur, "m_2 \otimes_A 1_M"'] & \end{tikzcd}$$

With the $X_2 \otimes_A (-)$-algebra structure induced by $m_2 \otimes_A 1_M$ this coequalizer is to be $f_*(M,\mu)$. Modulo the usual abuses of language, we should call this $X_2 \otimes_{X_1} M$.

The functor $f^b : X_2^b \longrightarrow X_1^b$ also has a right adjoint

$$f_+ : X_1^b \longrightarrow X_2^b$$

given as follows: let $(M,\tilde{\mu})$ be a $\mathrm{Hom}_A(X_1\ \ -\ )$-coalgebra; form the equalizer in A^b of the maps

$$\begin{tikzcd}[column sep=small]
\mathrm{Hom}_A(X_2, M) \arrow[rr, "\mathrm{Hom}_A(1_{X_2}, \tilde{\mu})"] \arrow[dr, "(\tilde{m}_2)_M"'] & & \mathrm{Hom}_A(X_2, \mathrm{Hom}_A(X_1, M)) \\
& \mathrm{Hom}_A(X_2, \mathrm{Hom}_A(X_2, M)) \arrow[ur, "\mathrm{Hom}_A(1_{X_2}, \mathrm{Hom}_A(f, 1_M))"'] &
\end{tikzcd}$$

With the $\mathrm{Hom}_A(X_2, -\ )$-coalgebra structure induced from $(\tilde{m}_2)_M$ this equalizer is to be $f_+(M,\mu)$, or in sloppy, but more suggestive language, $\mathrm{Hom}_{X_1}(X_2, M)$.

Now we construct a functor from A-Alg to A-theories. If X is an A-algebra, let us temporarily denote by $\overline{X}$ the algebraic structure of

$$X^b \xrightarrow{e^b} A^b \xrightarrow{U_A} \underline{S} \quad .$$

In view of proposition 6.6., X^b is an algebraic category with $X^b \xrightarrow{e^b} A^b \xrightarrow{U_A} \underline{S}$ as forgetful functor, so we may write $\overline{X}^b = X^b$ and $U_{\overline{X}} = U_A . e^b$. Further, we have a map of theories

$$A \xrightarrow{\bar{e}} \overline{X}$$

such that $\bar{e}^b = e^b$. From (9.1) we deduce that for every map of A-algebras $X_1 \xrightarrow{f} X_2$, we have a commuting diagram of theories

$$\begin{tikzcd}
& A \arrow[dl, "\bar{e}_1"'] \arrow[dr, "\bar{e}_2"] & \\
\overline{X}_1 \arrow[rr, "\bar{f}"] & & \overline{X}_2
\end{tikzcd}$$

where $\bar{f}^b = f^b$. In this way, we get a functor

$$(X, m, e) \longrightarrow (A \xrightarrow{\bar{e}} \overline{X}), \quad f \longrightarrow \bar{f}$$

from Alg(A) to the category of A-theories. We may ask, when is an A-theory in the essential image of this functor, i.e. when is an A-theory $A \xrightarrow{e} B$ isomorphic to one arising from an A-algebra? Clearly, a necessary condition is that e^b should have a right adjoint. We will show that this condition is also sufficient.

Theorem 9.2 A necessary and sufficient condition that a map of theories $A \xrightarrow{f} B$ should arise from an A-algebra is that f^b should have a right adjoint.

Proof. The necessity is clear. For sufficiency, suppose f^b has a right adjoint f_+. Then the composite

$$A^b \xrightarrow{f_*} B^b \xrightarrow{f^b} A^b$$

has a right adjoint, namely the composite

$$A^b \xrightarrow{f_+} B^b \xrightarrow{f^b} A^b .$$

So, by theorem 8.5., the composite $f^b f_*$ is of the form $X \otimes_A (-)$ for a unique (A,A)-bimodel X. The monad structure of $f^b f_*$ makes X into an A-algebra. Proposition 6.6 ensures that $B^b = X^b$, and $B = \overline{X}$.

This suggests that we should call a map of theories f essential if f^b has a right adjoint as well as a left adjoint.

Theorem 9.3 The functor $X \longmapsto \overline{X}$ is an equivalence between the category of A-algebras and the category of theories essential over A.

Proof. All that remains to be shown is that the functor $X \longmapsto \overline{X}$ is full. Suppose that X_1, X_2 are A-algebras and that

$$\begin{tikzcd} & A \arrow[dl, "\bar{e}_1"'] \arrow[dr, "\bar{e}_2"] & \\ \bar{X}_1 \arrow[rr, "g"] & & \bar{X}_2 \end{tikzcd}$$

is a commutative diagram of theories. Let

$$u : 1_{X_1^b} \longrightarrow g^b g_*$$

be the front adjunction of the adjoint pair (g_*, g^b). Then

$$e_1^b u e_{1*} : X_1 \otimes_A (-) \longrightarrow X_2 \otimes_A (-)$$

is a map of monads, and so arises from a unique map of A-algebras

$$h : X_1 \longrightarrow X_2 .$$

In order to show that $g = \bar{h}$, we must show that $g^b = h^b$. If (M,μ) is an X_2-module, then

$$h^b(M,\mu) = (m,\mu . h \otimes_A 1_M) = (M,\mu . e_1^b u e_{1*}) = g^b(M,\mu),$$

where, for the last equality, we have freely used the equivalence of A-models with $U_A F_A$-algebras.

In view of this result we shall drop the bar and denote the A-theory $\bar{X}$ simply by X, as is the fashion in the annular case.

If X is an A-algebra, let us look a little more closely at the structure of X-models. Let us describe an X-model by its $\mathrm{Hom}_A(X, -)$-coalgebra structure: so we are given an A-model M and a homomorphism of A-models

$$M \xrightarrow{\tilde{\mu}} \mathrm{Hom}_A(X, M),$$

satisfying certain conditions. Consider the function

$$U_A(M) \xrightarrow{U_A(\tilde{\mu})} U_A(\mathrm{Hom}_A(X, M)) = \mathrm{Hom}_{Ab}(U_{[A,A]}(X), M).$$

Given an element $x \in U_A . U_{[A,A]}(X)$ and an element $y \in U_A(M)$, we may evaluate $U_A(U_A(\tilde{\mu})(y))$ on x to get an element $x.y$ in $U_A(M)$. In this way, the elements x of the underlying set of X act on the underlying sets of X-models - they give unary operations of the theory X, in fact.

We can see this from the identity

$$\Omega_1(X) \simeq U_A . U_{[A,A]}(X) .$$

On the left hand side X is interpreted as a theory, on the right hand side as an A-algebra.

Suppose now that $w \in \Omega_S(A)$ and that M has an X-model structure. Using the identity

$$U_A(\mathrm{Hom}_A(X,M)) = \mathrm{Hom}_{Ab}(U_{[A,A]}(X),\ M)$$

and

$$U_A(\mathrm{Hom}_A(X(A_S),M)) = \mathrm{Hom}_{Ab}(\coprod_S U_{A,A}(X), M) =$$

$$= \prod_S \mathrm{Hom}_{Ab}(U_{[A,A]}(X),\ M),$$

we obtain a commutative diagram

$$\begin{tikzcd}
\prod_S U_A(M) \arrow[r, "\prod_S \tilde{\mu}"] \arrow[d, "M(\omega)"'] & \mathrm{Hom}_{Ab}(\coprod_S U_{[A,A]}(X), M) \arrow[d, "\mathrm{Hom}_{Ab}(X(\omega), M)"] \arrow[dd, bend left=70, "\ulcorner x.\omega \urcorner"] \\
U_A(M) \arrow[r, "\tilde{\mu}"] \arrow[dr, "x.(-)"'] & \mathrm{Hom}_{Ab}(U_{[A,A]}(X), M) \arrow[d, "\ulcorner x \urcorner"] \\
 & U_A(M)
\end{tikzcd}$$

where $\ulcorner x \urcorner$ denotes "evaluate at x" and x.w. denotes the image of x under

$U_A(X(w))$. The commutativity of the diagram gives us that if z is an S-indexed family of elements of $U_A(M)$ then

$$x.(w.z) = (x.w).z \ .$$

This may be interpreted as a sort of distributivity law as will be seen in a moment. The theory X may be thought of as an extension of A by unary operations (the elements of the underlying set of X) some of which are identified with unary operations of A (via the homomorphism of monoids $\Omega_1(A) \longrightarrow \Omega_1(X)$ induced by $A \xrightarrow{e} X$) which obey the distributivity law just mentioned.

An instructive example is the following: let A be the theory of commutative rings, and let X be the theory of commutative rings with derivation. There is an obvious map of theories $A \longrightarrow X$, which is essential. We obtain X by adjoining a unary operation d satisfying the distributivity laws $d.0 = d.1 = 0$, $d(x+y) = d.x + d.y$

$$d(x.y) = dx.y + x.dy \ .$$

Another example of a theory essential over (commutative rings) is the theory of special λ-rings. An example of a distributivity law is the Cartan formula

$$\lambda^n(x+y) = \sum_{p+q=n} \lambda^p(x) \ . \ \lambda^q(y).$$

Exercises 9

1. Show that annular theories are precisely those essential over $\mathbb{Z}$ and that unary theories are precisely those essential over $\underline{S}$.

2. In the example of the essential map of theories (comm. rings) $\longrightarrow$ (comm. rings with derivation) describe in detail the corresponding (comm. rings)-algebra.

3. Show that the forgetful functor

$$\text{A-Alg} \longrightarrow [A, A]$$

has a left adjoint, and construct it, proceeding by analogy with the notion of tensor algebra.

4. Show that the unique map of theories $A \longrightarrow 1$ is essential if and only if A has precisely one nullary operation.

5. Show that A-Alg has pushouts.

§10. Commutative theories

Commutative rings have many nice properties; for example, if R is a commutative ring and M and N are R-modules, the set $\mathrm{Hom}_R(M,N)$ has a natural R-module structure; also we may define the tensor product $M \otimes_R N$ which is again an R-module. We shall generalize the notion of commutativity to theories, and we shall see that there are results which generalize these mentioned above.

In §1 we defined the concept of commutation for two operations. We repeat the definition here:-

Definition 10.1 Let A be an algebraic theory, S and T sets, and let $\alpha \in \Omega_S(A)$ and $\beta \in \Omega_T(A)$. We say that α and β commute if for any $S \times T$-indexed family $\{x_{\sigma\tau}\}$ of elements of $U_A(X)$, for any A-model X,

$$\alpha^{\sigma} \beta^{\tau} x_{\sigma\tau} = \beta^{\tau} \alpha^{\sigma} x_{\sigma\tau} .$$

Of course, we could rewrite this definition is an element free way using $\delta^{S \times T}_{(\sigma,\tau)}$'s instead of $x_{\sigma,\tau}$'s. In informal language, we may say that α and β commute if for any $S \times T$-matrix of elements the following two processes give the same result:-

i) apply α to the rows to obtain a $1 \times T$ column. Then apply β to this.

ii) apply β to the columns to get an $S \times 1$ row. Then apply α to this.

The following points should be noted:-

1) If β is a nullary operation, then α commutes with β if and only if

$$A_1 \xrightarrow{\alpha} A_S \longrightarrow A_1 \xrightarrow{\beta} A \quad = A_1 \xrightarrow{\beta} A$$

i.e. if $\alpha(\beta, \ldots, \beta) = \beta$.

2) Two nullary operations commute if and only if they are equal.

3) A unary operation always commutes with itself.

4) An S-ary operation does not necessarily commute with itself unless the cardinality of S is less than 2. For example, if α is a binary operation, it commutes with itself if and only if it satisfies the law

$$\alpha(\alpha(x_{11},x_{12}),\alpha(x_{21},x_{22})) = \alpha(\alpha(x_{11},x_{21}), \alpha(x_{12},x_{22})).$$

This is the so called "entropic law"

The relation "α commutes with β" sets up a polarity on subsets of operations of a theory. We extend this to maps of a theory by saying that two maps

$$A_S \xrightarrow{\alpha} A_{S'} \quad , \quad A_T \xrightarrow{\beta} A_{T'}$$

of a theory A commute if $\delta_\sigma^S \alpha$ commutes with $\delta_\tau^T \beta$ for all $\sigma \in S$ and $\tau \in T$. If H is any collection of maps of A, we define the <u>commutant</u> H^c of H by

$$H^c = \{\alpha \in A \mid \alpha \text{ commutes with } \beta \text{ for all } \beta \in H\}.$$

<u>Theorem 10.2</u> For any subclass H of maps of A, H^c is a subtheory of A.

The proof of 10.2 is trivial and we omit it. As with all polarities we have

$$H_1 \subseteq H_2 \implies H_2^c \subseteq H_1^c$$

$$H \subseteq H^{cc} \quad , \quad H^c = H^{ccc} .$$

We denote A^c by $Z(A)$ and call it the centre of A. If $A = Z(A)$ we say that A is a commutative theory. These definitions agree with the conventional ones in the unary and annular cases.

Lemma 10.3 Let $A_S \xrightarrow{\alpha} A_T$ be a map in A. Then α belongs to the centre of A if and only if for every A-model X, the action

$$X(\alpha) \ : \ X(A_T) \longrightarrow X(A_S)$$

is the underlying function of a homomorphism $\prod_T X \longrightarrow \prod_S X$.

The proof is an immediate corollary of the discussion in §1 about homomorphisms being functions which commute with operations.

We say that a category C is enriched over a theory B if $\mathrm{Hom}_C : C^o \times C \to \underline{S}$ factors through $U_B : B^b \longrightarrow \underline{S}$.

Theorem 10.4 $Z(A)$ is the largest subtheory B of A such that A^b is enriched over B.

Proof. First we show that A^b is enriched over $Z(A)$, i.e. that Hom_{A^b} factors through $U_{Z(A)}$.

Let $w \in \Omega_S(A)$ and let

$$\{ f_\sigma : X \longrightarrow Y \}_{\sigma \in S}$$

be an S-indexed family of homomorphisms.

Let

$$\nabla : X \longrightarrow \prod_S X$$

denote the diagonal homomorphism, and let $w^\sigma f_\sigma$ denote the composite

$$X \xrightarrow{\nabla} \prod_S X \xrightarrow{\prod f_\sigma} \prod_S Y \xrightarrow{Y(w)} Y .$$

If $w \in Z(A)$ this is a homomorphism, and in this way $\mathrm{Hom}_{A^b}(X,Y)$ carries a $Z(A)$-model structure which we denote by $\mathrm{Hom}_A(X,Y)$. It is clearly natural in X and Y, and Hom_A gives the required enrichment. Conversely, if A^b is enriched over $B \subseteq A$, then by lemma 10.3, $B \subseteq Z(A)$.

<u>Corollary 10.5</u> If A is a commutative theory, there is a functor

$$\mathrm{Hom}_A : A^{b^o} \times A^b \longrightarrow A^b$$

such that $\quad U_A . \mathrm{Hom}_A = \mathrm{Hom}_{A^b}$.

Suppose that M and N are A-models and that $w \in \Omega_S(Z(A))$. The isomorphism

$$\mathrm{Hom}_{A^b}(\coprod_S M, N) \simeq \prod_S \mathrm{Hom}_{A^b}(M,N) \simeq \mathrm{Hom}_{A^b}(M, \prod_S N)$$

together with the action $\mathrm{Hom}_A(M,N)(w)$ give us a natural map

$$\mathrm{Hom}_A(\coprod_S M, -) \longrightarrow \mathrm{Hom}_A(M, -)$$

which, by the Yonede lemma, must arise from a homomorphism

$$M \longrightarrow \coprod_S M$$

which we call the <u>co-action</u> of w on M. In this way, every A-model has a natural co-$Z(A)$-model structure in A^b, i.e. a $(Z(A),A)$-bimodel structure. We obtain a functor

$$A^b \longrightarrow \left[Z(A), A \right]$$

which splits the forgetful functor $[Z(A), A] \longrightarrow A^b$, and so is full and faithful. From the functor

$$\otimes_{Z(A)} \; : \; [Z(A), A] \times Z(A)^b \longrightarrow A^b$$

we get a functor

$$A^b \times Z(A)^b \longrightarrow A^b$$

which we also denote by $\otimes_{Z(A)}$. When A is commutative we get a functor

$$\otimes_A \; : \; A^b \times A^b \longrightarrow A^b.$$

In this case, the adjointness of $\otimes_A$ and Hom_A can be enriched to a natural isomorphism

$$\mathrm{Hom}_A(M \otimes_A N, L) \simeq \mathrm{Hom}_A(N, \mathrm{Hom}_A(M, L)).$$

Let us consider a concrete elementwise construction for $M \otimes_A N$. Let F be the free A-model generated by symbols $(\bar{m}, \bar{n})$ for $m \in U_A(M)$, $n \in U_A(N)$. Let Γ be the congruence on F generated by the elements

$$((w^{\sigma} \bar{m}_{\sigma}, \bar{n}), \; w^{\sigma}(\bar{m}_{\sigma}, \bar{n}))$$
$$((\bar{m}, w^{\sigma} \bar{n}_{\sigma}), \; w^{\sigma}(\bar{m}, \bar{n}_{\sigma}))$$

for $w \in \Omega_S(A)$, where $\{m_{\sigma}\}$, $\{n_{\sigma}\}$ are S-indexed families of elements of $U_A(M)$, $U_A(N)$ and $m \in U_A(M)$. We let $m \otimes n$ denote the image of $(\bar{m}, \bar{n})$ in F/Γ. This construction clearly gives the right universal property and closely parallels the construction of tensor product of modules. Its advantage to us is to demonstrate the following:-

<u>Proposition 10.6</u> If A is a commutative theory, there is a coherent natural isomorphism

$$t \; : \; M \otimes_A N \longrightarrow N \otimes_A M : m \otimes n \longrightarrow m \otimes n.$$

Thus, if A is commutative, A^b is a closed symmetric monoidal category.

Exercises 10

1. Show that the centre of (Groups) is the theory of pointed sets.

2. Let M and N be models of an affine commutative theory. Show that $M \cdot N$ is a quotient of $M \otimes_A N$.

3. Let A be a commutative theory and let X be an A-algebra, which as an (A,A)-bimodel is in the image of $A^b \longrightarrow [A,A]$. Call X commutative if the diagram

$$\begin{tikzcd} X \otimes_A X \arrow[rr, "t"] \arrow[dr, "m"'] & & X \otimes_A X \arrow[dl, "m"] \\ & X & \end{tikzcd}$$

commutes.

Show that X is commutative as a theory if and only if X is commutative as an A-algebra.

4. Formulate conditions on a monad on $\underline{S}$ in order that its associated theory should be commutative.

5. For what theories is the category of models Cartesian closed (i.e. for what A does the functor $X \times (-) : A^b \longrightarrow A^b$ have a right adjoint, for every $X \in A^b$)?

§11. Free Theories

Theories are usually described, in practice, in terms of certain generating operations and laws between them. It is clear that an algebraic theory is itself a sort of algebraic gadget, but a many-sorted one. Instead of one forgetful functor we have a whole collection of functors Ω_S, one for each cardinality. As has been already pointed out, theories and maps of theories do not form a legitimate category. The category $B_\alpha th$ of α-bounded theories and maps between them does, and so indeed does their direct limit

$$Bth = \varinjlim_{\alpha} B_\alpha th$$

the category of bounded theories.

Let $Card_\alpha$ denote the category of families of sets and functions indexed by the cardinals less than α. This is a legitimate category. If $\alpha \leqslant \beta$ we have a full and faithful functor $Card_\alpha \longrightarrow Card_\beta$ which inserts an object of the former in the latter category by assigning to all the indices greater than or equal to α the empty set. Let

$$Card = \varinjlim_{\alpha} Card_\alpha .$$

This is the category of cardinal indexed sets which are eventually empty. For each regular α, we define

$$\Omega^{(\alpha)} : B_\alpha th \longrightarrow Card_\alpha$$

by $A \longmapsto \{\Omega_S(A)\}$ where S ranges over cardinals less than α. In this chapter we wish to show that $\Omega^{(\alpha)}$ has a left adjoint

$$F^{(\alpha)} : \mathrm{Card}_\alpha \longrightarrow B_\alpha\mathrm{th} .$$

If $\alpha \leqslant \beta$, the diagram of functors

$$\begin{tikzcd} \mathrm{Card}_\alpha \arrow[r] \arrow[d, "F^{(\alpha)}"'] & \mathrm{Card}_\beta \arrow[d, "F^{(\beta)}"] \\ B_\alpha\mathrm{th} \arrow[r] & B_\beta\mathrm{th} \end{tikzcd}$$

will commute, so that we may pass to the limit and define

$$F = \varinjlim F^{(\alpha)} : \mathrm{Card} \longrightarrow \mathrm{Bth} .$$

It is theories isomorphic to those in the image of F that we shall call free.

Now we turn to the construction of free theories.

Let α be a regular cardinal, fixed for the rest of this section. Although the term "tree" has various more general interpretations, for us it will mean the following:

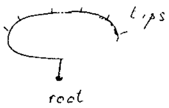
Definition A tree is a partially ordered set P satisfying the following conditions:

(i) P has a unique minimal element (the root),

(ii) the set $m(P)$ of maximal elements (the tips) has cardinality less than α,

(iii) for all $x \in P$, the subset $\{y / y \leqslant x\}$ is linearly ordered by the induced ordering from P.

In pictures:



For example, the following are trees:-

For any element x of a tree P, we denote by $\sigma(x)$ the set of minimal elements in $\{y \in P \mid x \leqslant y, x \neq y\}$. We call $\sigma(x)$ the branching set of x ; it consists of all the points lying immediately above x. Note that x is tip if and only if $\sigma(x) = \emptyset$.

A map between two trees is to be a monotone function. Thus we have a category of trees, and we may talk of trees being isomorphic. For example,

and

are isomorphic.

If P and Q are trees, and x is a tip of P, we can construct another tree $P \cup_x Q$, which we shall call the tree obtained by attaching Q to P at x. As a set, $P \cup_x Q$ is obtained by forming the disjoint union of P and Q and then identifying x with the root of Q.

The order relation on $P \cup_x Q$ is uniquely determined by requiring that the obvious inclusions

$$P \longrightarrow P \cup_x Q \qquad Q \longrightarrow P \cup_x Q$$

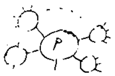
shall be monotone.

We may, of course, attach more than one tree at a time. So if P is a tree, and $\{Q_x\}_{x \in m(P)}$ is a family of trees indexed by the tips of P, we may form the tree

$$P \cup \{Q_x\}_{x \in m(P)}$$

obtained by attaching Q_x to x, for each tip x of P. Notice that we need α to be a regular cardinal in order to ensure that the resulting partially ordered set is still a tree. Note also that we have a bijection

$$\coprod_{x \in m(P)} m(Q_x) \simeq m(P \cup \{Q_x\}_{x \in m(P)}).$$

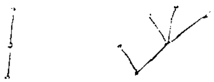


Now we define an important subclass of trees by an inductive method. A tree is of <u>type 1</u> if every element is either a tip or a root. For example,

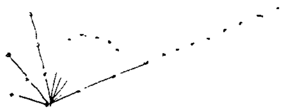
are of type 1. Note that trees isomorphic to trees of type 1 are themselves of type 1.

For any ordinal β, a tree is of <u>type β</u> if it can be obtained by attaching trees of type γ, for $\gamma < \beta$, to a tree of type 1, and if β is minimal with this property.

Thus



are of type 2, while



is of type ω, because it is obtained by attaching trees of finite type to a tree of type 1 (with a countable number of tips).

Again, note that a tree isomorphic to one of type β is of type β. We call a tree <u>regular</u> if it is of type β for some ordinal β.

An infinite linearly ordered chain

is not regular (if it were, suppose it was of type β ; remove the bottom link, and we must get a tree of type less than β, a contradiction).

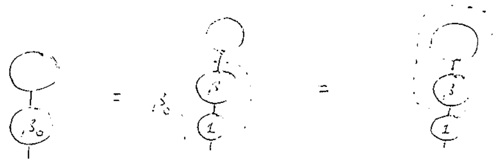
Proposition 11.1 The branching set of any element of a regular tree has cardinality less than α.

Proof. We use induction on type. It is clearly true for trees of type 1. Suppose it is true for all trees of type γ for all $\gamma < \beta$. Then it is true for a tree of type β because every element of such a tree either belongs to a subtree of type γ for $\gamma < \beta$, or is the root, in which case it belongs to a subtree of type 1.

This principle of expressing any regular tree as obtained from a tree of type 1 by sticking on trees of lower type is fundamental. In some sense of the word, regularity is a condition on trees which uniformly bounds the height of the tree, but not its breadth.

Theorem 11.2 Trees obtained by attaching regular trees to the tips of a regular tree are regular.

Proof: Let P be a tree of type β, and let $\{Q_i\}_{i \in m(p)}$ be a family of trees indexed by the set of tips of P, where Q_i is of type δ_i. We must show that $P \cup \{Q_i\}_{i \in m(p)}$ is regular. We do this by transfinite induction on β. Let δ be the least ordinal greater than the ordinals δ_i, $i \in m(P)$. If $\beta = 1$ then $P \cup \{Q_i\}_{i \in m(P)}$ has type δ. Now suppose the theorem proved for all β less than β_0. If P has type β_0, then it is obtained by attaching trees of type β for $\beta < \beta_0$ to a tree of type 1. Hence $P \cup \{Q_i\}_{i \in m(P)}$ is obtained by attaching to a tree of type 1 trees which are regular by the inductive hypothesis. In pictures



Basically, what we have used is the associativity of the attaching process. Now we are back to case $\beta = 1$, and the induction is complete.

Suppose that X_* is an object of Card_α. The basic idea of the construction of $F^{(\alpha)}(X_*)$, the free theory on X_*, is to represent the operations of the theory by trees, with nodes labelled by elements of X_*. There is a natural way in which we can think of an operation as a tree; imagine that we wish to evaluate an operation w on an s-indexed family of elements $\{x_\sigma\}_{\sigma \in S}$, where of course w has arity S. Represent w by a tree with tips in bijective correspondence with S, and imagine x_σ stuck on the σ-th tip (in the charming nomenclature of R. Vogt, the x 's are cherries, so we have a cherry tree), then "passing down the tree" we come to $w^\lambda x_\lambda$ at the root

It is clear that composition of operations corresponds precisely to the notion of attaching trees.

If P is a regular tree, an <u>X_*-labelling of arity S</u> on P is an assignment ϵ of a value of $X_{\sigma(x)}$ to each element x of $P - S$, where S is a subset of $m(P)$, whose elements we call the <u>free tips</u> of (P,ϵ). In virtue of proposition 11.1 this makes sense.

We say that two X_*-labelled regular trees of arity S, (P,ϵ) and (P',ϵ') are isomorphic if there exists an isomorphism

$$f : P \longrightarrow P'$$

such that $\epsilon'(f(x)) = \epsilon(x)$ for all $x \in P$, which identifies the corresponding sets of free tips.

Now we may describe how to attach labelled regular trees to each other. Suppose that (P,ϵ) is an X_*-labelled regular tree of arity S, and that for each $i \in S$, (Q_i,γ_i) is an X_*-labelled regular tree of arity V_i. Then we define an X_*-labelled regular tree of arity $\coprod_{i\in S} V_i$

$$(R,\zeta) = (P,\epsilon) \cup \{(Q_i,\gamma_i)_{i\in S}\}$$

by $R = P \cup \{Q_i\}_{i\in S}$, where $\zeta|Q_i = \gamma_i$ and $\zeta|(P-S) = \epsilon$.

Note that attaching labelled trees preserves the relation of of isomorphism.

We can now describe the elements of $\mathcal{D}_S(F^{(\alpha)}(X_*))$ as pairs

$$(f, [P,\epsilon])$$

where $T \xrightarrow{f} S$ is a function, T is a set of cardinality less than α, and $[P,\epsilon]$ is an isomorphism class of X_*-labelled regular trees of arity T. Now we must describe how maps in $F^{(\alpha)}(X_*)$ compose. For this purpose we write

$$w = (f, [P, \ell]) \quad \text{where } T \xrightarrow{f} S$$

$$\alpha_\sigma = (g_\sigma, [Q_\sigma, \psi_\sigma]) \quad \text{where } T_\sigma \xrightarrow{g_\sigma} S_\sigma$$

where $\sigma \in S$.

We must describe $w\langle\alpha_*\rangle$ as $(h, [R, \xi])$: for each $t \in T$, attach a copy of $(Q_{f(t)}, \psi_{f(t)})$ to the tip t of P, and call the resulting X_*-labelled regular tree (R, ξ). It has arity

$$\coprod_{\sigma \in S} (T_\sigma \times f^{-1}(\sigma)) .$$

We define the function

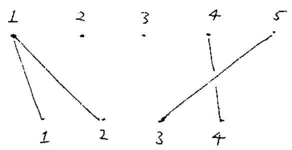
$$h : \coprod_{\sigma \in S} (T_\sigma \times f^{-1}(\sigma)) \longrightarrow \coprod_{\sigma \in S} S_\sigma$$

by the formula $h(t', t) = g_{f(t)}(t')$.

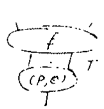
In order to understand the reason for these formulae it is necessary to get a better picture of the operations $(f, [P, \ell])$. We already know how to picture (P, ℓ) as a tree with all but some of its tips labelled (the labelled tips correspond to nullary operations of course). The purpose of the function f is to permute, omit or repeat the "variables". Imagine f as a bundle of lines, where for each $\sigma \in S$ we draw a line from σ to each element of $f^{-1}(\sigma)$; or conversely, from each element of T to its image in S.

Example: $f : \{1,2,3,4\} \longrightarrow \{1,2,3,4,5\}$

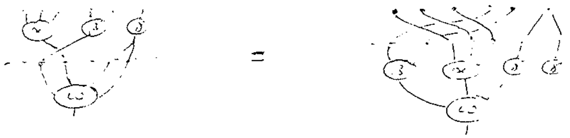
$f(1) = 1$, $f(2) = 1$, $f(3) = 5$, $f(4) = 4$.



Picture $(f, [P, \varepsilon])$ as



The whole point about composition of operations is that lines can be "disentangled", so that the functionlike parts get pushed to the top, e.g.

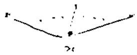


I hope the above picture makes clear the reason for the formulae given above. In fact, pictures are not only more informative than words, they give quicker proofs. In terms of pictures, the associativity of composition is almost immediate. In symbols it is most tedious.

It is immediate from the definition that $F^{(\alpha)}(X_*)$ is α-bounded. Note that the map $\dot{\sigma}^S$ in $F^{(\alpha)}(X_*)$ is given by $(\dot{\sigma}, [*, \emptyset])$ where $\dot{\sigma} : 1 \longrightarrow S$ is the insertion of σ, $*$ is a one element tree and $\emptyset$ denotes the empty labelling.

If $G : X_* \longrightarrow Y_*$ is a map in Card_α, we get a map of theories $F^{(\alpha)}(G) : F^{(\alpha)}(X_*) \longrightarrow F^{(\alpha)}(Y_*)$ just by relabelling, so the construction gives a functor.

If $x \in X_S$, let us denote by $\bar{x}$ the type 1 tree with root labelled by x, with tips in bijective correspondence with S:

$\bar{x}$ = 

We define a map

$$\eta(X_*) : X_* \longrightarrow \Omega_* (F^{(\alpha)}(X_*))$$

in Card_α by assigning to $x \in X_S$ the element $(1_S, [\bar{x}])$. It is not hard to see that this gives a natural map

$$\eta : 1_{\mathrm{Card}_\alpha} \longrightarrow \Omega_* F^{(\alpha)}$$

which is to serve as the front adjunction.

We point out that in $F^{(\alpha)}(X_*)$, the map $(f, [P, \theta])$ can be expressed as the composite

$$1_T, [P, \theta]). F^{\alpha}(X_*)_f$$

where $T \xrightarrow{f} S$. It follows that any map of algebraic theories with domain $F^{(\alpha)}(X_*)$ is uniquely determined by its values on maps of the form $(1_T, [P, \theta])$. But every such element can be expressed as a composite

$$(1_{T'}, [P', \theta']). < 1_{V_*}, [Q_*, \psi_*] >$$

where P' is of type 1, and each Q_i is of type less than that of P. Hence, by induction, maps with domain $F^{(\alpha)}(X_*)$ are uniquely determined by their values on maps in the image of $\eta(X_*)$. So we define the end adjunction

$$\epsilon : F^{(\alpha)}\Omega_* \longrightarrow 1_{B_\alpha th}$$

by requiring that $\epsilon(A) : F^{(\alpha)}(\ _*(A)) \longrightarrow A$ be given by $\epsilon(A)\,(1_S, [\bar{x}]) = x$.

It is now straightforward to verify that η and $\mathcal{E}$ furnish front and end adjunctions making $F^{(\alpha)}$ left adjoint to Ω_*.

It is interesting to note that although each functor

$$F^{(\alpha)} : \mathrm{Card}_\alpha \longrightarrow \mathrm{B}_\alpha\mathrm{th}$$

has a right adjoint, the direct limit functor

$$F : \mathrm{Card} \longrightarrow \mathrm{Bth}$$

does not. Nevertheless, every object of Bth has a semisimplicial resolution by free theories, and up to homotopy this resolution is functorial. For a similar construction for topological algebraic theories see BOARDMAN AND VOGT .

Exercises 11

1. Show that if $\alpha = \aleph_0$, then regular trees are finite, and every map in a free finitary theory is a finite composite of generating operations (i.e. trees of type 1).

2. Show that if $\alpha > \aleph_0$ it is not necessarily the case that every map in a free α-bounded theory is a finite composite of generating operations.

3. If $X_* = (\emptyset, X_1, \emptyset, \ldots) \in \mathrm{Card}_2$, show that $\Omega_1 F^{(2)}(X_*)$ is the free monoid on X_1.

4. Show that subtheories of free theories are not necessarily free.

Hint; consider the free monoid on one generator t, and the submonoid generated by t^2 and t^3.

§ 12. Completeness of the category of bounded theories.

Algebraic theories are themselves algebraic gadgets, but of a many sorted type. That is to say, we may consider the functors Ω_S as a collectively faithful family of forgetful functors.

We have already remarked that if A and B are algebraic theories, the class of maps from A to B may not be a set if A is unbounded. To obtain a legitimate category we can restrict our attention to bounded theories; of course, many of the constructions we perform may work for certain unbounded theories for reasons particular to the special cases in question. Apart from problems of boundedness, the construction of limits and colimits of theories proceeds in a manner strictly analogous to the constructions of § 4. For this reason we will merely sketch the outlines, leaving tedious verification to the incredulous reader.

The construction of free theories shows that the functors Ω_S must preserve (and reflect) limits. So if

$$\{A_i\}$$

is a diagram of theories (not necessarily bounded) we define a theory B by the formula

$$\mathrm{Hom}_B(B_T, B_S) = \varprojlim_i \mathrm{Hom}_{A_i}(A_{i_T}, A_{i_S}).$$

Since products commute with limits, B is a theory. The canonical projections define maps $B \longrightarrow A_i$ making B the limit of the diagram $\{A_i\}$ in the illegitimate category of theories.

Let us interpret the notion of product of theories in terms of models. Suppose that $\{A_i\}_{i \in I}$ is a family of theories, indexed by a set I. The functor

$$\prod_{i \in I} A_i^b \xrightarrow{\prod_{i \in I} U_{A_i}} \prod_{i \in I} \underline{S} \xrightarrow{\prod} \underline{S} \qquad (12.1)$$

has a left adjoint

$$\underline{S} \xrightarrow{\nabla} \prod_{i \in I} \underline{S} \xrightarrow{\prod_{i \in I} F_{A_i}} \prod_{i \in I} A_i^b$$

where $\prod$ is the functor taking an I-indexed family of sets $\{V_i\}_{i \in I}$ to $\prod_{i \in I} V_i$ and ∇ is the diagonal functor taking a set S to the constant family of sets, each equal to S. This adjoint pair defines a monad on $\underline{S}$ whose algebraic theory is clearly $\prod_{i \in I} A_i$ (look at the Kleisli category). But (12.1) is tripleable, so we may identify

$$\left(\prod_{i \in I} A_i\right)^b \quad \text{with} \quad \prod_{i \in I} A_i^b$$

with (12.1) as forgetful functor. Thus an $\prod_{i \in I} A_i$ - model is, as a set, just a product of A_i-models, and a homomorphism is a function which is a product of homomorphisms of A_i-models.

If p is the projection from $\prod_{i \in I} A_i$ to A_j, then p_* is the projection from $\prod_{i \in I} A_i^b$ to A_j^b and p^b is the functor which takes an A_j-model X to the family of A_i-models, $\{Y_i\}$ where $Y_i = 1$ for $i \neq j$, and $Y_j = X$.

We have already defined, in § 5, the notion of subtheory. If $A \xrightarrow{f} B$ is a map of theories, then the image Imf of the functor f is a subtheory of B, and the induced map $A \longrightarrow$ Imf, makes Imf a quotient theory of A. Quotient theory is another concept we defined in § 5 .

Just as for models, we define a <u>congruence</u> Γ on a theory A to be an equivalence relation on A, i.e. a subtheory of $A \times A$ such that for every set S, $\Omega_S(\Gamma)$ is an equivalence relation on $\Omega_S(A)$.

Note that unless A is bounded, the class of congruences on A may not be a set. In any case, an intersection of congruences on a theory is a congruence, so we may talk of the congruence generated by a class of pairs of operations. If Γ is a congruence on A we may define the quotient theory A/Γ, just as we did for models. For any set S, $\Omega_S(A/\Gamma)$ is $\Omega_S(A)/\Omega_S(\Gamma)$. Using congruences we can now construct coequalizers of maps of theories. The coequalizer of

$$A \overset{f}{\underset{g}{\rightrightarrows}} B$$

is $B \longrightarrow B/\Gamma$ where Γ is the congruence on B generated by the pairs $(f(\omega), g(\omega))$ as ω ranges over all operations of any arity of A.

For any map $A \xrightarrow{f} B$, we denote by Kerf the kernel pair of f, i.e.

$$\begin{tikzcd} \mathrm{Kerf} \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

is a pullback diagram. Then Kerf is a congruence on A, and $A \longrightarrow A/\mathrm{Kerf}$ is the coequalizer of $\mathrm{Kerf} \rightrightarrows A$. We have the usual factorization theorem that A/Kerf is canonically isomorphic to Imf.

In order to construct coproducts we have to restrict ourselves to bounded theories, at least for the methods we outline here. Coproducts of unbounded theories cannot exist in general; for example, the theory CH of question 3 ex. 2 has no coproduct with the free theory on one unary operation.

Suppose that $\{A_i\}$ is a family of bounded theories. By the results of the previous chapter we may write

$$A_i = F(V_{i*})/\Gamma_i$$

where V_{i*} is an object of Card, and Γ_i is a congruence on the free theory $F(V_{i*})$. Then

$$\coprod_i A_i = F(\coprod_i V_{i*})/\Gamma$$

where Γ is the congruence generated by the images of the congruences Γ_j under the maps

$$F(V_{j*}) \longrightarrow F(\coprod_i V_{i*})$$

induced by the inclusions into the coproduct in Card.

In chapter 7 we saw that "semantics", the functor assigning to a theory A the functor $A^b \xrightarrow{U_A} \underline{S}$, had a left adjoint "algebraic structure", which was also a left inverse. Hence "semantics" preserves and reflects colimits in Bth to limits in the category of categories over $\underline{S}$.

Suppose that $\{A_i\}$ is a family of bounded theories. The limit of the functors $U_{A_i} : A_i^b \longrightarrow \underline{S}$ is their joint pullback over $\underline{S}$. We may interpret the domain of this functor as the category of families $\{X_i\}$, where $X_i \in A_i^b$, where the sets $U_{A_i}(X_i)$ are all the same. That is to say, a $\coprod_i A_i$ - model is a set which simultaneously has A_i-model structures for each i. A homomorphism of $\coprod_i A_i$-models is a function which is a homomorphism of A_i-models for each i. If

$$p : A_j \longrightarrow \coprod_i A_i$$

is the canonical injection, then p^b is the functor which assigns to the object $\{X_i\}$ in $(\coprod_i A_i)^b$ the A_j-model X_j. The description of p_* is rather more complicated.

Similarly, if

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "k"] & C \end{tikzcd}$$

is a coequalizer diagram of theories,

$$\begin{tikzcd} C^b \arrow[r, "k^b"] & B^b \arrow[r, shift left, "f^b"] \arrow[r, shift right, "g^b"'] & A^b \end{tikzcd}$$

will be an equalizer diagram of categories. We know in any case that functors of the form f^b are faithful. The interpretation of the equalizer diagram is that C^b is isomorphic to the subcategory of B^b of those B-models and homomorphisms of B-models on which f^b and g^b agree. This is a full subcategory, because the diagram

$$\begin{tikzcd} C^b \arrow[r, "k^b"] & B^b \arrow[r, shift left, "f^b"] \arrow[r, shift right, "g^b"'] \arrow[d, "U_B"'] & A^b \arrow[dl, "U_A"] \\ & \underline{S} & \end{tikzcd}$$

commutes and U_A is faithful.

As a corollary we have

<u>Proposition 12.2.</u> If the map of theories $A \xrightarrow{f} B$ makes B into a quotient theory of A, then $f^b : B^b \longrightarrow A^b$ is full and faithful.

We can prove this in an alternative way that requires no boundedness conditions on A or B. If

$$\eta : 1_{A^b} \longrightarrow f^b f_*$$

is the front adjunctio[illegible]r the adjoint pair (f_*, f^b), then for each set S we have a commutative diagram

$$\begin{tikzcd} \Omega_S(A) \arrow[r, "\Omega_S(f)"] \arrow[d, "\wr"] & \Omega_S(B) \arrow[d, "\wr"] \\ U_A F_A(S) \arrow[r, "U_A \eta F_A S"] & U_B F_B(S) \end{tikzcd}$$

So if $\Omega_S(f)$ is surjective, $f^b f_* F_A(S)$ is a quotient model of $F_A(S)$. But every model is a colimit of free models, and a colimit of regular epics is a regular epic. Hence η_X is a regular epic for all A-models X. But the adjunction identity

$$f^b\varepsilon \cdot \eta f^b = 1_{f^b}$$

implies that ηf^b is monic. If it is also a regular epic, it is an isomorphism, and hence f^b is also an isomorphism. Since f^b reflects isomorphisms, the end adjunction

$$f_* f^b \xrightarrow{\varepsilon} 1_{B^b}$$

is an isomorphism, and so f^b is full and faithful.

The interpretation of proposition 12.2. is clear. If B is a quotient theory of A it is obtained from A by adding more laws. Its models are precisely those A-models which satisfy the extra laws.

In the other direction we have:

<u>Proposition 12.3.</u>

Let $A \xrightarrow{f} B$ be a map of bounded theories for which $f^b : B^b \longrightarrow A^b$ is full and faithful. Then f is epic in Bth.

<u>Proof.</u> Note that to say that f is epic does not imply that B is a quotient theory of A. We have the famous example of the inclusion $\mathbb{Z} \subseteq \mathbb{Q}$, where $\mathbb{Q}$ denotes the ring of rationals. This is an epic map both in the category of rings and in the category of theories.

Let $B \underset{A}{*} B$ denote the cokernel pair of $A \xrightarrow{f} B$, i.e. we have a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d] \\ B \arrow[r] & B \underset{A}{*} B \end{tikzcd} .$$

We have a canonical codiagonal map $B \underset{A}{*} B \xrightarrow{\Delta} B$. The map f is epic if and only if Δ is an isomorphism, if and only if $\Delta^b : B^b \longrightarrow (B \underset{A}{*} B)^b$ is an isomorphism. Now a $(B \underset{A}{*} B)^b$-model, as we have seen above, is a pair (X, Y) of B-models, such that $f^b X = f^b Y$. The functor Δ^b is given by $\Delta^b(Z) = (Z, Z)$. Now if f^b is full and faithful, $f^b X = f^b Y$ implies that $X = Y$, and so Δ^b is an isomorphism.

The methods of [Stenström] page 77 generalize to prove that f^b is full when f is essential and $B \otimes_A B \longrightarrow B$ is an isomorphism.

Isbell has pointed out that the inclusion of theories

$$A \xrightarrow{f} B$$

where A is the theory generated by an associative binary operation, and B is the theory of monoids, is epic, but that f^b is not full. For example, consider the A-model given by two generators e, u satisfying

$$e^2 = e, \quad u^2 = u, \quad eu = ue = e.$$

The endomorphism φ given by $\varphi(u) = \varphi(e) = e$ is not in the image of f^b, even though the A-model in question clearly has a B-model structure.

Every bounded algebraic theory is a coequalizer of a pair of maps between free theories. To see this, let A be a bounded theory. By the previous chapter, A can be written as F_0/Γ_0 where F_0 is a free theory and Γ_0 is a congruence on F_0. Now we may express Γ_0 as F_1/Γ_1 in like manner. The composites $F_1 \twoheadrightarrow F_1/\Gamma_1 \xrightarrow{\sim} \Gamma_0 \rightarrowtail F_0 \times F_0$ give a pair of maps $F_1 \rightrightarrows F_0$ of which A is the coequalizer. We call a coequalizer diagram

$$F_1 \rightrightarrows F_0 \longrightarrow A$$

with F_0, F_1 free theories a <u>presentation of</u> A. In practice this is the usual way of describing theories. The generators of F_0 are the "primitive operations", the generators of F_1 are the "axioms".

A presentation is the low dimensional part of a semi-simplicial free resolution

$$\cdots\; F_2 \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longleftarrow} \; F_1 \; \substack{\longrightarrow \\ \longrightarrow \\ \longleftarrow} \; F_0 \longrightarrow A$$

where the generators of F_2 represent the relations between the axioms, and so on. We shall not investigate this notion further, but will linger only to remark that we have passed a signpost to a very major side road indeed, with little traffic on it as yet. We refer the reader to [Beck 3, Boardman and Vogt, Stasheff].

Exercises 12.

1. Show that a map of theories f is a regular epic if and only if $\Omega_S(f)$ is surjective for all S.

2. In Bth show that pullbacks of regular epics are regular epics.

3. Show that for a fixed regular cardinal α, the pair of adjoint functors

$$B_\alpha th \underset{\longrightarrow}{\overset{F^{(\alpha)}}{\longleftarrow}} Card_\alpha$$

makes $B_\alpha th$ tripleable over $Card_\alpha$.

4. Show that the functor

$$A\text{-Alg} \longrightarrow A\text{-theories}$$

preserves and reflects colimits.

5. For every A-model X construct a functor

$$\tilde{X} : (A\text{-theories}) \longrightarrow (X, A^b)$$

(see ex. 7 question 2) which assigns to $A \xrightarrow{f} B$ the front adjunction

$$X \xrightarrow{\eta} f^b f_* X.$$

Show that the functors $F_A\widetilde{(S)}$ are collectively faithful, and that they preserve and reflect limits and regular epics.

6. Show that if the A-model X is not free, the functor $\tilde{X}$ may not preserve limits.

§ 13. The Kronecker product

If A and B are bounded theories we denote their coproduct by $A * B$. Let us denote the canonical injections by

$$A \xrightarrow{i_1} A * B, \quad B \xrightarrow{i_2} A * B$$

(they are not necessarily monic, of course). We may consider the congruence Γ on $A * B$ generated by pairs of the form $(i_1(\alpha)^{\sigma} i_2(\beta)^{\tau},\ i_2(\beta)^{\tau}\ i_1(\alpha)^{\sigma})$ for all operations α of A and β of B (the dummy superfixes σ and τ are supposed to indicate that the operations act on $S \times T$-indexed families of elements). We denote $A*B/\Gamma$ by $A \otimes B$. Informally, $A \otimes B$ is the theory with operations of both A and B, the axioms that hold in A and B separately, and axioms saying that A-operations commute with B-operations, and no other independent axioms. From the previous chapter we see that $A \otimes B$-models are sets with an A-model structure and a B-model structure, such that the A-operations are B-homomorphisms or vice-versa. It follows that $(A \otimes B)^b$ is simply the category of A-models in B^b, or equivalently, of B-models in A^b.

We may consider the full subcategory $\mathrm{CBth} \subseteq \mathrm{Bth}$ of commutative bounded theories. The inclusion functor has a left adjoint

$$A \longrightarrow A/\Gamma_A$$

where Γ_A is the congruence on A generated by pairs $(\alpha^{\sigma} \beta^{\tau},\ \beta^{\tau} \alpha^{\sigma})$ for all operations α, β of A. Thus A/Γ_A is A "made commutative".

<u>Proposition 13.1.</u> If A and B are commutative bounded theories then $A \otimes B \simeq A*B/\Gamma_{A*B}$. In particular $\otimes$ is coproduct in CBth.

The proof is a straightforward argument about congruences which we omit.

Theorem 13.2. Let A be a finitary theory, and let B be an annular theory. Then $A \otimes B$ is an annular theory.

Proof. Let α be the image in $A \otimes B$ of an n-ary operation of A, and let + and 0 be the images in $A \otimes B$ of addition and zero in B. That α commutes with + can be stated by the equality

$$\alpha(x_1+x_1', \ldots, x_n+x_n') = \alpha(x_1, \ldots, x_n) + \alpha(x_1', \ldots, x_n')$$

for any 2n-ple $(x_1, \ldots, x_n, x_1', \ldots, x_n')$ of elements of an $A \otimes B$-model. That α commutes with 0 can be written

$$\alpha(0, \ldots, 0) = 0.$$

It follows from these two identities, by successively taking some of the variables to be zero, that

$$\alpha(x_1, \ldots, x_n) = \alpha_1(x_1) + \ldots + \alpha_n(x_n)$$

where $\alpha_1(x) = \alpha(x, 0, \ldots, 0)$, $\alpha_2(x) = \alpha(0, x, 0, \ldots 0)$, etc. Now each of the α_j's is a B-linear unary operation, and any extension of a ring by linear unary operations is a ring.

In particular, for any finitary theory A, $A \otimes \mathbb{Z}$ is a ring, A finitary theory A is annular if and only if $A \longrightarrow A \otimes \mathbb{Z}$ is an isomorphism. The full and faithful functor

$$(\text{Rings})^b \longrightarrow \text{Bth}$$

has a left adjoint, which we might call annulization

$$A \longrightarrow A_{fin} \otimes \mathbb{Z}$$

where A_{fin} is the finitary part of A.

Theorem 13.3. Let A be the theory generated by a binary operation with a two-sided identity. Then $A \otimes A$ is isomorphic to the theory generated by an associative commutative binary operation with a two-sided identity.

Proof. Let α_i, e_i be the images of the binary operation, two-sided identity respectively, for the two factors, for $i = 1, 2$. Since e_1 commutes with e_2 we have $e_1 = e_2 = e$ say. Since α_1 commutes with α_2 we have

$$\alpha_1(\alpha_2(x_{11}, x_{12}), \alpha_2(x_{21}, x_{22})) = \alpha_2(\alpha_1(x_{11}, x_{21}), \alpha_1(x_{12}, x_{22}))$$

for any x_{11}, x_{12}, x_{21}, x_{22} in an $A \otimes A$-model. The substitution $x_{12} = x_{21} = e$ gives $\alpha_1 = \alpha_2 = \alpha$, say. The substitution $x_{11} = x_{22} = e$ gives $\alpha(x,y) = \alpha(y,x)$, so that α is commutative. The substitution $x_{21} = e$, gives $\alpha(\alpha(x,y),z) = \alpha(x, \alpha(y,z))$, so α is associative. We refer the reader to [Hilton].

Corollary 13.4. $(Gp) \otimes (Gp) \simeq \mathbb{Z}$.

Suppose that G is a monoid, and that A is a theory. Because in $[A, A]$ we have $\coprod_G A \otimes_A \coprod_G A \simeq \coprod_{G \times G} A$ the structure maps $G \times G \longrightarrow G$, $1 \longrightarrow G$ determine an A-algebra structure for $\coprod_G A$.

Proposition 13.5. As a theory $\coprod_G A \simeq A \otimes G$. The models are A-models with an action of G as a monoid of endomorphisms.

Proof. Let (M, μ) be an algebra of the monad $\coprod_G A$ on A^b. Then

$$\coprod_G M \simeq (\coprod_G A) \otimes_A M \xrightarrow{\mu} M$$

determines, for each $g \in G$, an endomorphism of M

$$M \xrightarrow{i_g} \coprod_G M \longrightarrow M$$

where i_g is the canonical injection. The usual axioms that μ must satisfy give a homomorphism of monoids

$$G \longrightarrow \mathrm{Hom}_{A^b}(M,\ M).$$

In exercises 12, number 4, we have remarked that the inclusion functor

$$A\text{-Alg} \longrightarrow (A\text{-theories})$$

preserves and reflects colimits. In exercises 10, number 3, we have defined the notion of a commutative algebra over A. We leave it to the reader to check that the theory associated to a commutative A-algebra is commutative, and that the usual tensor product of (A,A)-bialgebras, i.e. $\otimes_A$, gives coproduct of commutative A-algebras. Thus, if

$$A \longrightarrow B\,, \qquad A \longrightarrow C$$

are maps of theories, if $A \longrightarrow B \otimes_A C$ stands for the coequalizer of

$$\begin{tikzcd}[row sep=small] & B \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & B \otimes C \\ & C \arrow[ur] & \end{tikzcd}$$

then in the algebraic case, our use of the symbol $\otimes_A$ is entirely consistent.

<u>Proposition 13.6.</u> Consider the pushout diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "\delta"] \\ C \arrow[r, "\gamma"] & B \otimes_A C \end{tikzcd}$$

in the category of commutative theories, algebraic over A. Then there are natural isomorphisms

i) $\beta^b \alpha_* \simeq \gamma_* \delta^b$

ii) $\alpha^b \beta_* \simeq \delta_* \gamma^b$

iii) $\beta^b \alpha_+ \simeq \gamma_+ \delta^b$

iv) $\alpha^b \beta_+ \simeq \delta_+ \gamma^b$.

Proof. It is only necessary to prove i) as the others follow by symmetry and taking adjoints. We may rewrite i) as the "cancellation" law,

$$(B \otimes_A C) \otimes_C X \simeq B \otimes_A X$$

for $X \in C^b$. It is enough to show that we have an isomorphism when X is free.

Exercises 13

1. If $A \xrightarrow{f} B$ is an algebraic map of theories, show that B is a quotient of $A * \Omega_1(B)$.

2. If A, B, C are bounded theories establish an equivalence between $[A \otimes B, C]$ and the category of coproduct preserving functors $A \longrightarrow [B, C]$.

3. If A is a theory and $S \subseteq \Omega_1(A)$, show that there exists a theory $A[S^{-1}]$ and a map of theories $A \xrightarrow{p} A[S^{-1}]$ such that every map of theories out of A which takes operations in S to isomorphisms factors uniquely through p.

4. Show that $A \xrightarrow{p} A[S^{-1}]$ is epic, and that p^b is full and faithful.

5. Describe a functor $A \longrightarrow A_{fin}$ (see [Beck, 3]).

§ 14. Extensions

We call a map of bounded theories

$$f : A \longrightarrow B$$

an extension if it can be factored

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & B \\ & A \otimes C \arrow[ur, "p"'] & \end{tikzcd}$$

where i is the injection into the Kronecker product (not necessarily monic) and p is a regular epic, i.e. $\Omega_S(p)$ is surjective for all S. Without loss of generality we may assume that C is free.

Equivalently, f is an extension if B is generated by $\mathrm{Im} f$ and by a set of operations, each member of which commutes with all the operations of $\mathrm{Im} f$. If C itself may be taken to be commutative, we call the extension central.

The following propositions are immediate consequences of the definition:

Proposition 14. 1. A composite of (central) extensions is a (central) extension.

Proposition 14. 2. For any bounded theory. A, the inclusion

$$Z(A) \hookrightarrow A$$

is an extension.

Proposition 14. 3. If $f: A \longrightarrow B$ is an extension, then $f(Z(A)) \subseteq Z(B)$.

Suppose that X is an (A,A)-bimodel. The elements of $U_A(U_{[A,A]}(X))$ are in bijective correspondence with A-model homomorphisms

$$F_A(1) \simeq U_{[A,A]}(A) \longrightarrow U_{[A,A]}(X).$$

We denote by $P(X)$ the subset of $U_A(U_{[A,A]}(X))$ corresponding to those A-model homomorphisms which underly homomorphisms of (A,A)-bimodels

$$A \longrightarrow X.$$

Thus, we set $P(X) \simeq \mathrm{Hom}_{[A,A]}(A,X)$. An element of $P(X)$ we call a primitive element. If A is an annular theory, an (A,A)-bimodel X is given by an (A,A)-bimodule; an element $x \in X$ is primitive, if for all elements a of the ring A, $ax = xa$. Let us look at another example : if A = (commutative rings), and X is an (A,A)-bimodel, with co-addition $\alpha : X \longrightarrow X \otimes_{\mathbb{Z}} X$, comultiplication $: X \longrightarrow X \otimes_{\mathbb{Z}} X$, counit $\epsilon : X \longrightarrow \mathbb{Z}$ and cozero $0 : X \longrightarrow \mathbb{Z}$, then $x \in X$ is primitive if

$$\alpha(x) = x \otimes 1 + 1 \otimes x$$

$$(x) = x \otimes x$$

$$e(x) = 1$$

$$0(x) = 0.$$

Because the elements of $P(X)$ are homomorphisms, they commute with all the operations of A. It follows that $P(X)$ has a natural $Z(A)$-model structure given as follows: if $\omega \in \Omega_S(Z(A))$ and $\{A \xrightarrow{f_\sigma} X\}_{\sigma \in S}$ is an S-indexed family of maps in $[A,A]$, we define $\omega^\sigma f_\sigma$ to be the composite

$$A \xrightarrow{A(\omega)} \coprod_S A \xrightarrow{\coprod_{\sigma \in S} f_\sigma} \coprod_S X \xrightarrow{\nabla} X$$

where ∇ is the codiagonal map. This is a homomorphism of (A,A)-bimodels because

ω commutes with all operations of A. In this way we get a functor

$$P : [A, A] \longrightarrow Z(A)^b.$$

We say that an (A,A)-bimodel X is primitively generated if $U_{[A,A]}(X)$ is generated as an A-model by the primitive elements of X. We denote by $\{A\}$ the full subcategory of $[A,A]$ of primitively generated (A,A)-bimodels.

Proposition 14.4. An (A,A)-bimodel X is primitively generated if and only if there is a set S and a map of (A,A)-bimodels

$$\coprod_S A \xrightarrow{p} X$$

such that $U_A(U_{[A,A]}(p))$ is surjective.

Note that S may be taken to be $U_{Z(A)}P(X)$.

Corollary 14.5. If X and Y are primitively generated (A,A)-bimodels, so is $X \otimes_A Y$.

Proof. $\otimes_A$ preserves "surjections".

Proposition 14.6. If the map of theories $A \xrightarrow{f} B$ is algebraic, then B as an $[A,A]$-bimodel is primitively generated if and only if f is an extension.

Proof. An element of $U_A(U_{[A,A]}(B))$ determines a unary operation of B. This operation commutes with operations of Imf if and only if it is a primitive element.

Theorem 14.8. Let $A \xrightarrow{f} B$ be an extension. Then there exists a unique functor

$$f_! : \{A\} \longrightarrow \{B\}$$

such that the diagram

$$
\begin{tikzcd}
\{A\} \arrow[rr, "f_!"] \arrow[d, hook] & & \{B\} \arrow[d, hook] \\
{[A,A]} \arrow[d, "U_{[A,A]}"'] & & {[B,B]} \arrow[d, dashed, "U_{[B,B]}"] \\
A^b \arrow[rr, "f_*"] & & B^b
\end{tikzcd}
$$

commutes.

<u>Proof.</u> It is enough to prove the theorem for two special cases of f :-

(i) $A \xrightarrow{f} B$ is regular epic, i.e. B is a quotient theory of A. Let $X \in \{A\}$, and let $\coprod_S A \xrightarrow{p} X$ be such that $U_A(U_{[A,A]}(p))$ is surjective.

Consider the diagram

$$
\begin{tikzcd}
 & & A \arrow[rrr, "f"] \arrow[dd, "\coprod_S A"] \arrow[dd, bend right=70] & & & B \arrow[dd, "\coprod_S B"] \arrow[dd, dashed, bend right=50] \\
X & \arrow[l, Rightarrow, "p"'] & & & f_!(X) & \arrow[l, dashed, Rightarrow, "f_!(p)"'] \\
 & & A^b \arrow[rrr, "f_*"] & & & B^b
\end{tikzcd}
$$

We want to construct a functor $f_!(X) : B \longrightarrow B^b$ which preserves coproducts, so that $f_* . X = f_!(X) . f$, and a natural map of regular epics

$$f_!(p) \;:\; \coprod_S B \longrightarrow f_!(X).$$

We must have $f_!(X)(B_S) = f_*(X(A_S))$, so $f_!(X)$ is determined on objects. Now we turn our attention to how $f_!(X)$ should be determined on maps of B. Let $B_U \xrightarrow{\beta} B_V$ be a map in B. Since f is regular epic, there exists $A_U \xrightarrow{\alpha} A_V$ in A such that $f(\alpha) = \beta$.

Now we consider the commuting diagram

$$\begin{tikzcd} \coprod_S F_A(U) \arrow[r, "\coprod_S I_A(\alpha)"] \arrow[d, "p_{A_U}"'] & \coprod_S F_A(V) \arrow[d, "p_{A_V}"] \\ X(A_U) \arrow[r, "X(\alpha)"] & X(A_V) \end{tikzcd}$$

Applying the functor f_* to this diagram, and using the identity

$$f_*\left(\coprod_S I_A(\alpha)\right) = \coprod_S I_B(f(\alpha)) = \coprod_S I_B(\beta)$$

we get the commutative diagram

$$\begin{tikzcd} \coprod_S F_B(U) \arrow[r, "\coprod_S I_B(\beta)"] \arrow[d, "f_*(p_{A_U})"'] & \coprod_S F_B(V) \arrow[d, "f_*(p_{A_V})"] \\ f_*(X(A_U)) \arrow[r, "f_*(X(\alpha))"] & f_*(X(A_V)) \end{tikzcd}$$

Since f_* preserves regular epics, $f_*(p_{A_U})$ is epic, and so $f_*(X(\alpha))$ is independent of the choice of α. So we set

$$f_!(X)(\beta) = f_*(X(\alpha))$$

$$f_!(p)_{B_U} = f_*(p_{A_U})$$

which clearly defines $f_!(X)$ and $f_!(p)$ uniquely. If

$$\theta : X \longrightarrow Y$$

is a map in $\{A\}$, a similar argument shows that $f_!(\theta)$ must be given by $f_!(\theta)_{B_U} = f_*(\theta_{A_U})$.

(ii) For the second case, we suppose that

$$A \xrightarrow{f} A \otimes C$$

is the canonical injection for the first factor, where C is a free theory. An $A \otimes C$-model may be described as a pair (M,μ) where M is an A-model, and μ denotes an action of the generators of C as A-model homomorphisms. Maps in $(A \otimes C)^b$ may be described as A-model homomorphisms which commute with the given actions. Clearly $f^b(M,\mu) = M$. If X is an (A,A)-bimodel, we may define a functor

$$(A \otimes C)^b \longrightarrow (A \otimes C)^b \; : \; (M,\mu) \longrightarrow (\mathrm{Hom}_A(X, M), \ \mathrm{Hom}_A(X,\mu)).$$

Since

$$\mathrm{Hom}_{(A \otimes C)^b}(f_* U_{[A,A]}(X), \ (M,\mu)) \simeq \mathrm{Hom}_{A^b}(U_{[A,A]}(X), \ M)$$

we see that the above functor is represented by an $(A \otimes C, A \otimes C)$-bimodel whose underlying $A \otimes C$-model is $f_* U_{[A,A]}(X)$. We denote the representing $(A \otimes C, A \otimes C)$-bimodel by $f_!(X)$. The fact that the action μ commutes with homomorphisms of A-models gives us, by use of the Yoneda lemma, that a map of (A,A)-bimodels $X \longrightarrow X'$ lifts to a map of $(A \otimes C, A \otimes C)$-bimodels $f_!(X) \longrightarrow f_!(X')$. Note that we have actually proved more than we need, for we did not assume that X was primitively generated. It is immediate that $f_!(\coprod_S A) = \coprod_S (A \otimes C)$ and that $f_!$ preserves "surjections". Hence $f_!$ takes primitively generated (A,A)-bimodels to primitively generated $(A \otimes C, A \otimes C)$-bimodels, and so defines a functor

$$\{A\} \longrightarrow \{A \otimes C\}.$$

Since every extension can be factored

$$A \longrightarrow A \otimes C \longrightarrow B$$

where the first map is as in case (ii) and the second as in case (i), we have proved the theorem.

It is not true in general that a map of theories $A \longrightarrow B$ induces a functor $[A,A] \longrightarrow [B,B]$. The theorem above tells us that we have such a functor if we restrict ourselves a) to maps of theories which are extensions, and b) to primitively generated bimodels. The necessity of the latter condition is not hard to understand. Once one has defined a map on the generators of a primitively generated bimodel, to check that one has a map of bimodels it is enough to ensure that it takes the generators to primitive elements. Readers who are used to manipulations with Hopf algebras will recognize this phenomenon.

If A is a commutative theory, it is clear that the canonical inclusion $A^b \hookrightarrow [A,A]$ factors through $\{A\}$. Let B be an arbitrary theory, and let $j : Z(B) \hookrightarrow B$ be the inclusion map.

<u>Theorem 14.8.</u> The composite

$$Z(B)^b \hookrightarrow \{Z(B)\} \xrightarrow{j_!} \{B\} \hookrightarrow [B,B]$$

is left adjoint to

$$P : [B,B] \longrightarrow Z(B)^b.$$

The proof is simply a corollary of the remarks above.

It may be interesting to study those theories B for which

$$j_! : \{Z(B)\} \longrightarrow \{B\}$$

is an equivalence of categories. We cite as examples annular theories which are Azumaya algebras, and (Gp).

Exercises 14.

1. Let A be a theory, X an A-algebra. Show that a primitive element of X corresponds to a unary operation of X which commutes with the images of the operators of A under $A \longrightarrow X$.

2. In the algebraic extension

$$A = (\text{rings}) \longrightarrow (\text{rings with a derivation}) = X$$

find all the primitive elements of X.

3. Show that if every primitively generated (A,A)-bimodel is a colimit of free (A,A)-bimodels then the functor

$$j_! : \{Z(A)\} \longrightarrow \{A\}$$

is an equivalence of categories.

4. Show that the conditions of ex. 3 above hold if for every set S_0 and subfunctor T of

$$\prod_{S_0} : A^b \longrightarrow A^b$$

which has a left adjoint, there is a set S_1 and a pair of natural maps

$$\prod_{S_0} \rightrightarrows \prod_{S_1}$$

whose equalizer is T.

§ 15. Morita equivalence and Matrix theories.

We say that two theories A and B are Morita equivalent if there is an equivalence of categories $A^b \simeq B^b$. From theorem 8.5. it follows that in that case there exist $X \in [A,B]$, $Y \in [B,A]$ such that

$$X \otimes_A Y \simeq B \quad \text{and} \quad Y \otimes_B X \simeq A.$$

It follows that

$$\mathrm{Hom}_B(X, -) \simeq Y \otimes_B (-)$$
$$\mathrm{Hom}_A(Y, -) \simeq X \otimes_A (-)$$

since inverse equivalences are adjoint. From chapter 7 we see that $U_{[A,B]}(X)$ and $U_{[B,A]}(Y)$ are regular projective generators.

Evaluating the above natural equivalences at X and Y respectively, we get

$$A \simeq \mathrm{Hom}_B(X,X) \quad , \quad B \simeq \mathrm{Hom}_A(Y,Y)$$

and $$Y \simeq \mathrm{Hom}_B(X,B) \qquad X \simeq \mathrm{Hom}_A(Y,A).$$

Since the sets $\Omega_S(Z(A))$ and $\mathrm{Nat}(\Pi_S 1_{A^b}, 1_{A^b})$ are in bijective correspondence, it follows that Morita equivalent theories have isomorphic centres, i.e. $Z(A) \simeq Z(B)$. The functor $Z \longrightarrow X \otimes_A Z \otimes_A Y : [A,A] \longrightarrow [B,B]$ is clearly an equivalence of categories with inverse $K \longrightarrow Y \otimes_B K \otimes_B X$, and as this functor preserves free bimodels and "surjections" it specializes to give an equivalence $\{A\} \xrightarrow{\simeq} \{B\}$.

If V is a fixed set, and A is an algebraic theory, let us denote by $M_V(A)$ the full subcategory of A of all objects of the form $A_{V \times S}$. Clearly, $M_V(A)$ has coproducts, and every object in it is a coproduct of copies of A_V. It follows that $M_V(A)$ is an algebraic theory (but not a subtheory of A).

If $V = \{1, 2, \ldots, n\}$ and A is an annular theory, then $M_V(A)$ is again annular, and is in fact the theory associated to the ring of $n \times n$ matrices with coefficients in the ring A. For this reason, we call $M_V(A)$ a <u>matrix theory</u> over A. We have an obvious functor from theories to theories given by

$$A \longrightarrow M_V(A) \ , \quad (A \xrightarrow{f} B) \longmapsto (M_V(A) \xrightarrow{M_V(f)} M_V(B))$$

where $M_V(f)$ is the functor f restricted to the subcategory $M_V(A)$, with obvious natural equivalences

$$M_{V_1}(M_{V_2}(A)) \simeq M_{V_1 \times V_2}(A).$$

We also have a natural map

$$\delta_A : A \longrightarrow M_V(A)$$

given by $A_S \longrightarrow A_{V \times S}$, $\alpha \longrightarrow \coprod_S \alpha$, which corresponds in the annular case to the embedding of a ring into the subring of diagonal matrices.

We shall abbreviate $M_V(\underline{S})$ to simply M_V.

For any theory A, consider the pair of adjoint functors

$$A^b \underset{\prod_V}{\overset{\coprod_V}{\rightleftarrows}} A^b \quad .$$

The functor $\prod_V$ has a left adjoint and satisfies Beck's tripleability criterion, so there is an A-theory $A \xrightarrow{f} B$, with an isomorphism $B^b \xrightarrow{\sim} A^b$ such that

$$\begin{tikzcd} B^b \arrow[rr, "\sim"] \arrow[dr, "f^b"'] & & A^b \arrow[dl, "\prod_V"] \\ & A^b & \end{tikzcd}$$

and $f^b f_* = \prod_V \coprod_V$ as a monad on A^b. Thus

$\Omega_S(B) \simeq U_A \prod_V \coprod_V F_A(S) \simeq U_A \prod_V F_A(V \times S) \simeq \Omega_S(M_V(A))$

These bijections induce an isomorphism of A-theories

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "\delta_A"] & \\ B \arrow[rr, "\sim"] & & M_V(A) \end{tikzcd}$$

so we have discovered that $M_V(A)$-models may be interpreted as sets of the form $\prod_V M$, where M is the underlying set of an A-model, and that an $M_V(A)$-model homomorphism may be identified with a function $\prod_V h$ where h is the underlying function of a homomorphism of A-models.

Proposition 15.1. For any set V and theory A, A and $M_V(A)$ are Morita equivalent.

Corollary 15.2. $Z(M_V(A)) \simeq Z(A)$.

Proposition 15.3. For any set V and theory A,

$$M_V(A) \simeq A \otimes M_V \ .$$

Proof. An M_V-model is simply a set which is a V-th Cartesian power. But we have seen that A-models which are V-th cartesian powers are $M_V(A)$-models.

Corollary 15.4. For any two bounded theories A, B

$$M_V(A \otimes B) \simeq M_V(A) \otimes B.$$

Proposition 15.5. Any theory Morita-equivalent to $\underline{S}$ is of the form M_V for some set V.

Proof. Let $\Phi : \underline{S} \longrightarrow A^b$ be an equivalence of categories, and let V be a set such that

$$\Phi(V) \simeq F_A(1) .$$

Then $\bar{\Phi}(V\times S) \simeq \bar{\Phi}(\coprod_S V) \simeq \coprod_S \bar{\Phi}(V) \simeq \coprod_S F_A(1) \simeq F_A(S)$

Hence

$$\mathrm{Hom}_A(A_S, A_T) \simeq \mathrm{Hom}_{A^b}(F_A(S),\ F_A(T)) \simeq$$

$$\mathrm{Hom}_{A^b}(\bar{\Phi}(V\times S), \bar{\Phi}(V\times T)) \simeq \mathrm{Hom}_{\underline{S}}(V\times S,\ V\times T) \simeq$$

$$\mathrm{Hom}_{M_V}((M_V)_S,\ (M_V)_T).$$

This demonstrates an isomorphism $A \longrightarrow M_V$.

Exercises 15.

1. In the theory M_V, given a function $g : V \longrightarrow V$, let $\xi(g)$ be the V-ary operation which to a V-indexed family $\{x_i\}_{i \in V}$, $x_i = \{x_{ij}\}_{j \in V}$, of elements of an M_V-model $\prod_V X$, assigns the element $\{x_{i,g(i)}\}_{i \in V}$. Show that

(i) $\xi(1_V)$ is an affine operation,

(ii) If $\{h_k\}_{k \in V}$ is a V-indexed family of functions from V to V, then $\xi(g) < \xi(h_*) > = \xi(p)$ where $p(i) = h_{g(i)}(i)$.

2. With the notation of question 1, show that the theory generated by the operations $\xi(g)$ subject to axioms (i) and (ii) is isomorphic to M_V.

3. If $A \xrightarrow{f} B$ is an epic map of theories, show that

$$M_V(f) : M_V(A) \longrightarrow M_V(B)$$

Is also epic. Also show that $M_V(f)$ is monic when f is.

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