

Graphical algebraic foundations for monad stacks

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Higher-Order Programming with Effects
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Problem statement

? Effects in a pure language.

! Use monads.

? Monads don't compose.

! Use monad transformers (monad stacks).

? But which order...

$(\text{StateT } s . \text{ErrorT } e)$

vs

$(\text{ErrorT } e . \text{StateT } s)$

!? Current practice relies on:

Programmer insight and experience¹, and black art².

! Towards a systematic approach? Tool support?

¹HOPE reviewer #1.

²HOPE reviewer #3.

Demo (part I)

`http://www.cl.cam.ac.uk/~ok259/graphtool`

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<http://www.cl.cam.ac.uk/~ok259/graphtool>

Small print

Full details later, but the tool:

- ▶ can't handle all monad transformers; and
- ▶ might fail to find valid monad stacks.

Talk structure

1. Algebraic effects.
2. Cographs (aka series-parallel graphs).
3. Tool.
4. Conclusion.

Semantics for exceptions

Let m be a (set-theoretic) monad, and e a set of exceptions.

We say that $\langle m, \text{raise} \rangle$ is an e -exception monad if raise is a Kleisli arrow:

$$\text{raise} :: e \rightarrow m \emptyset$$

The *initial* e -exception monad is the exception monad $m\ a = \text{Error}\ e\ a$ with its standard raise operation.

I.e., for every other e -exception monad $\langle m', \text{raise}' \rangle$, there exists a unique monad morphism $h :: m \rightarrow m'$ satisfying for all $\text{exc} :: e$:

$$h(\text{raise}\ \text{exc}) = \text{raise}'\ \text{exc}$$

Algebraic effects (Plotkin and Power 2002)

Semantics for global state

Let m be a (set-theoretic) monad, and s a set of states.

We say that $\langle m, \text{get}, \text{put} \rangle$ is a *global s -state monad* if get and put are Kleisli arrows:

$$\text{get} :: () \rightarrow m\ s \quad \text{put} :: s \rightarrow m\ ()$$

such that the following three equations hold (Plotkin and Power 2002, and Mellies 2010):

$$\begin{array}{lll} x \leftarrow \text{get} (); & \text{put } x; & \text{put } x; \\ \text{put } x & = \text{return } () & \text{get } () = \text{return } x \end{array} \quad \text{put } x; \text{put } y = \text{put } y$$

(in $m\ ()$, $m\ s$, and $m\ ()$ respectively).

The *initial* global s -state monad is the global s -state monad $m\ a = \text{State } s\ a = s \rightarrow (s, a)$ with its get and put operations.

Algebraic effects (Plotkin and Power 2002)

Algebraic semantics (part 1)

A *presentation* is a triple $\langle \pi, ar, E \rangle$ consisting of a set π (of *generic effect symbols*) and a π -indexed collection of pairs of sets ar :

$$\langle p_{op}, a_{op} \rangle \quad op \in \pi$$

(the pair $\langle \pi, ar \rangle$ is called a *signature*), and E is a set of pairs of terms (called *equations*) involving the monadic *return* and *do* notation, and Kleisli arrows:

$$op :: p_{op} \rightarrow m \ a_{op}$$

for all $op \in \pi$.

Algebraic effects (Plotkin and Power n2002)

Algebraic semantics (part 2)

Given a presentation $P = \langle \pi, ar, E \rangle$, a P -monad is a monad m and an assignment of Kleisli arrows:

$$op :: p_{op} \rightarrow m \ a_{op}$$

for all $op \in \pi$, satisfying all the equations in E .

The *initial* P -monad m_P always exists.

All these concepts are well established and date back to Lawvere's thesis (1963) and to Linton (1966).

Plotkin and Power's algebraic theory of effects analyses monads used in the semantics of computational effects in terms of their presentations.

(Excludes the continuation monad, more details offline.)

Examples (Plotkin and Power 2002)

Previous examples

- ▶ Exceptions: $raise :: e \rightarrow m \emptyset$, no equations
- ▶ Global state: $get :: () \rightarrow m s$, $put :: s \rightarrow m ()$, as before

Additional examples

- ▶ Environment monad: $get :: () \rightarrow m s$, equations:

$$\begin{array}{lcl} x \leftarrow get (); & & x \leftarrow get (); \\ return () & = return () & y \leftarrow get (); = z \leftarrow get (); \\ & & return(x, y) \quad return(z, z) \end{array}$$

- ▶ Writer monad for a monoid $\langle mon, \cdot, 1 \rangle$: $act :: mon \rightarrow m ()$:

$$\begin{array}{lcl} act\ m_1; & & \\ act\ m_2 & = act\ (m_1 \cdot m_2) & act\ 1 = return() \end{array}$$

- ▶ Free monad for a functor F : no eqns (more details offline)

Examples

Additional examples (ctd)

- List monad: $fail :: () \rightarrow m \emptyset$, $choose :: () \rightarrow m \text{ bool}$
equations:

$$\begin{array}{l} x \leftarrow choose (); \\ \text{if } x \text{ then } fail \\ \text{else } return () \end{array} = return () = \begin{array}{l} x \leftarrow choose (); \\ \text{if } x \text{ then } return () \\ \text{else } fail \end{array}$$

$$\begin{array}{l} x \leftarrow choose(); \\ y \leftarrow choose(); \\ \text{case}(x, y) \text{ of} \\ \quad (True, True) \rightarrow return 1 \\ \quad (True, False) \rightarrow return 2 \\ \quad (_, False) \rightarrow return 3 \end{array} = \begin{array}{l} x \leftarrow choose(); \\ y \leftarrow choose(); \\ \text{case}(x, y) \text{ of} \\ \quad (True, _) \rightarrow return 1 \\ \quad (False, True) \rightarrow return 2 \\ \quad (False, False) \rightarrow return 3 \end{array}$$

Combining effects (Hyland, Plotkin, and Power 2006)

Sum

Every two presentations $P_1 = \langle \pi_1, ar_1, E_1 \rangle$, $P_2 = \langle \pi_2, ar_2, E_2 \rangle$ can be combined by the disjoint union of the operations $\pi_1 + \pi_2$, and subsequent relabelling of the equations. Call the resulting presentation their *sum*, denoted by $P_1 + P_2$.

Theorem

Let P_{exc} be the presentation for *e*-exceptions. For every presentation P :

$$m_{P_{\text{exc}}+P} \cong \text{ErrorT } e \ m_P$$

Therefore, the action of the exception monad transformer arises as the sum with the theory for exception.

Theorem

Let P_F be the presentation for the free monad for a functor F . For every presentation P : $m_{P_F+P} \cong \text{FreeT } F \ m_P$.

Combining effects (Hyland, Plotkin, and Power 2006)

Tensor

By adding the following equations

$$\begin{aligned}x_1 &\leftarrow op_1\ p_1 & x_2 &\leftarrow op_2\ p_2 \\x_2 &\leftarrow op_2\ p_2 & = & x_1 \leftarrow op_1\ p_1 \\return\ (x_1, x_2) && return\ (x_1, x_2)\end{aligned}$$

for all $op_1 \in \pi_1$, $op_2 \in \pi_2$ to the sum $P_1 + P_2$, we obtain another way to combine presentations, their tensor $P_1 \otimes P_2$.

Theorem

Let P_{st} , P_{env} , P_{mon} be the presentations for the s -state, s -environment and mon-writer monads. Then for every presentation P :

$$\begin{aligned}m_{P_{st} \otimes P} &\cong StateT\ s\ m_P & m_{P_{env} \otimes P} &\cong ReaderT\ s\ m_P \\m_{P_{mon} \otimes P} &\cong WriterT\ mon\ m_P\end{aligned}$$

Combining effects

Applicability

Covered the MTL (sans continuations).

Not all monad transformers arise as either sum or tensor, even when their associated monads arise from presentations.

Jaskelioff's ListT

$$\text{ListT } m \ a = m \ (\text{Either } () \ (a, \text{ListT } m \ a))$$

Theorem

Let P_{list} be the presentations for the list monad. For every presentation P : $m_{P_{\text{list}} \odot P} \cong \text{ListT } m_P$, where $P_{\text{list}} \odot P$ is obtained from $P_{\text{list}} + P$ by adding the following equation, for all $op \in \pi$:

$$\begin{array}{lcl} b \leftarrow \text{choose}(); & & y \leftarrow op \ p; \\ \text{if } b \text{ then } y \leftarrow op \ p; & & b \leftarrow \text{choose}(); \\ \quad \text{return Just } y & = & \text{if } b \text{ then return Just } y \\ \text{else return None} & & \text{else return None} \end{array}$$

Commutativity analysis (Hyland, Plotkin, and Power 2006)

Setting

Restrict attention to monad transformers arising as sum or tensor of theories (e.g., MTL).

Design choice

Choose, for every pair of effects, whether they should commute.

Analysis

Do these commutative equations:

- ▶ arise through sum and tensor of basic theories?
- ▶ result from a monad stack of the given transformers?

Commutativity analysis (Hyland, Plotkin, and Power 2006)

Setting

Restrict attention to monad transformers arising as sum or tensor of theories (e.g., MTL).

Design choice

Choose, for every pair of effects, whether they should commute.

$$\text{State} \bullet \bullet \text{Exceptions} \quad \text{vs} \quad \text{State} \bullet \leftrightarrow \bullet \text{Exceptions}$$

Analysis

Do these commutative equations:

- ▶ arise through sum and tensor of basic theories?

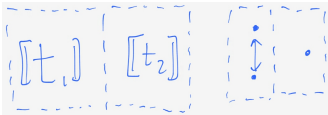
$$t ::= x \mid \sum_{i \in I} t_i \mid \bigotimes_{i \in I} t_i$$

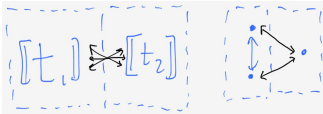
- ▶ result from a monad stack of the given transformers?

Graph theory (Hyland, Plotkin, and Power 2006)

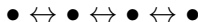
Every term denotes a graph:

$$\llbracket x \rrbracket = x \bullet$$

$$\llbracket t_1 + t_2 \rrbracket =$$


$$\llbracket t_1 \otimes t_2 \rrbracket =$$


But not all graphs arise in this way, e.g., P_4 :



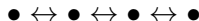
Cographs

Definition

A *cograph* is a graph isomorphic to $\llbracket t \rrbracket$ for some t (a.k.a. series-parallel graphs, ambiguously).

Theorem (Corneil et al. 1981)

A graph is a cograph $\iff P_4$ does not embed into it



- ▶ Witness for negative result.
- ▶ Polynomial time.
- ▶ Does not provide a sum and tensor decomposition.

Theorem (McConnell and Spinrad 1999)

There is a linear time algorithm for deciding whether a given graph is a cograph, and if so, exhibiting its sum and tensor decomposition.

- ▶ Computes the modular decomposition of the graph (more offline).
- ▶ Simpler algorithms in polynomial time.

Demo (part II)

`http://www.cl.cam.ac.uk/~ok259/graphtool`

Demo (part II)

<http://www.cl.cam.ac.uk/~ok259/graphtool>

Small print

- ▶ Only applies to algebraic effects (excludes continuations) arising as sum and tensor (excludes ListT).
- ▶ Might fail to find valid monad stacks.

$$\begin{array}{c} \text{non-determinism} + \text{exceptions} \\ = \\ \text{non-determinism} \otimes \text{exceptions} \end{array}$$

● ● $\leftrightarrow$ ● $\leftrightarrow$ ●

VS

● $\leftrightarrow$ ● $\leftrightarrow$ ● $\leftrightarrow$ ●

Conclusion

The algebraic perspective, regardless of the tool, is insightful.

Contributions

- ▶ Connecting this problem with cographs (suggested by Atkey).
- ▶ Characterising graphs arising from monad stacks (straightforward).
- ▶ The algebraic analysis of Jaskelioff's *ListT*.

Further work

- ▶ Beyond the MTL (e.g., Jaskelioff's thesis).
- ▶ No idea how to deal with continuations.